Scattering

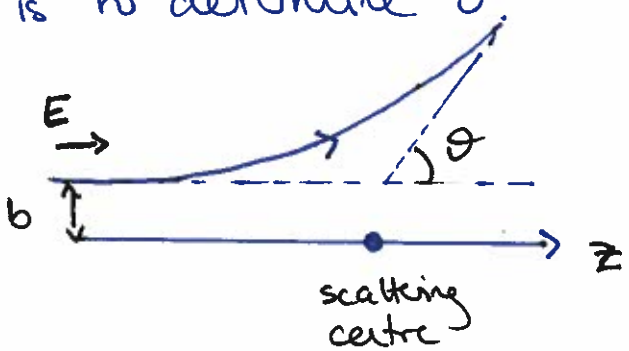
Chapter 10

Classical Scattering

Section 10.1

Before we study scattering in quantum systems, we'll take a look at scattering in classical systems (think: bowling). In particular, we'll consider classical scattering of a particle off a heavy target (so don't think bowling, think more of a marble scattering off a bowling ball)

The particle comes in with energy E and impact parameter b and the leaves at some angle ϑ. The whole point is to determine ϑ



> The impact parameter tells us how "head on" the collision is.

> so we can ignore the motion of the target

Example 10.1

Image a small pellet bounces elastically off a billiard ball, of radius R

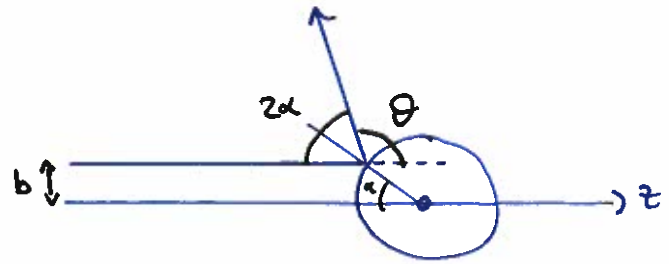
The impact parameter is

$b = R\sin\alpha$

But $2\alpha + \vartheta = \pi \Rightarrow \alpha = \frac{\pi - \vartheta}{2}$

$\Rightarrow b = R\sin\left(\frac{\pi}{2} - \frac{\vartheta}{2}\right) = R\cos\frac{\vartheta}{2}$

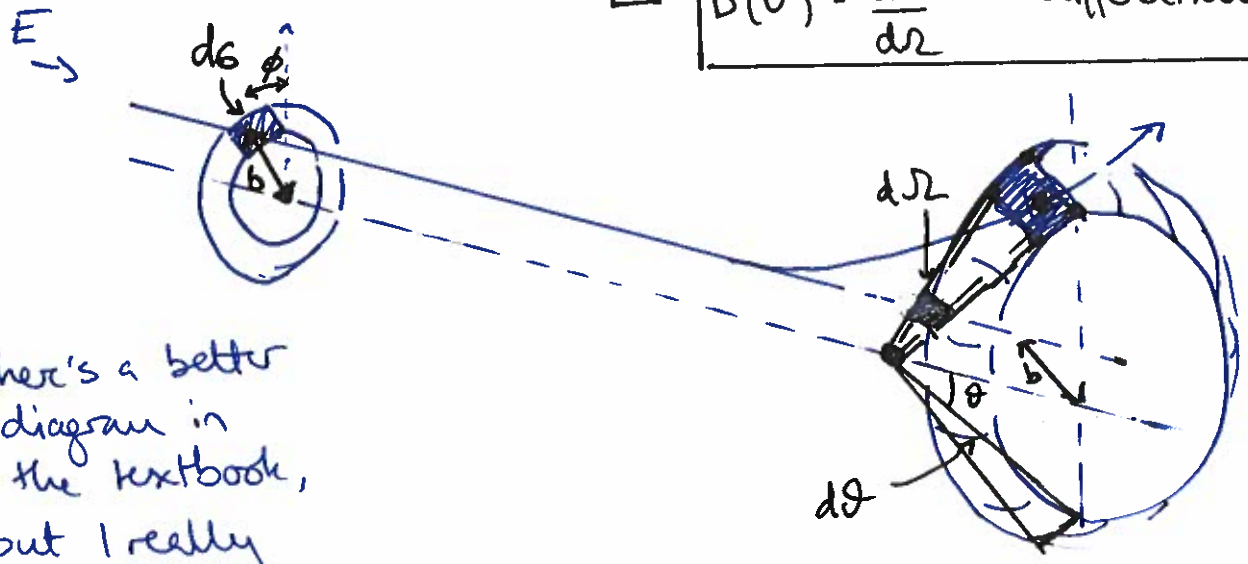
$\Rightarrow \vartheta = 2\cos^{-1}\left(\frac{b}{R}\right)$ if $b \leq R$ or $\vartheta = 0$ if $b > R$



In many cases, however, we're not scattering individual particles off targets, but a whole bunch of them, distributed within some small region $d\sigma$. These particles get scattered not into a single path, but into a patch around a given path. In other words the particles get scattered into a small solid angle $d\Omega$. Clearly, the larger $d\sigma$ is, the larger $d\Omega$ will be. The factor that relates them is called the differential (scattering) cross-section

$$d\sigma = D(\theta)\, d\Omega$$

$$D(\theta) = \frac{d\sigma}{d\Omega} \leftarrow \text{differential cross-section}$$



There's a better diagram in the textbook, but I really tried.

Now, the incident patch is given by

$$d\sigma = b\, db\, d\phi$$

And the outgoing patch is given by

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

Thus

$$D(\theta) = \frac{d\sigma}{d\Omega} = \frac{b \, db \, d\phi}{\sin\theta \, d\theta \, d\phi} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

The angle decreases as the impact parameter increases, so this derivative is usually negative; hence the absolute value signs.

The <u>total cross-section</u> is the integral of the differential cross-section

$$\sigma = \int D(\theta) \, d\Omega$$

Of course, we don't normally have a small circle of particles incident on a target - we have a <u>beam</u>. We can basically think of this as a cylinder, which we usually take to have a uniform density of particles.

How do we count the number of incident particles?

We imagine picking a reference point, and counting how many particles pass that point in a small interval of time dt.

The flux at that point is the number of particles passing per second. The luminosity is the flux per unit area

↑ denoted $\mathcal{L}$

> $\mathcal{L}$ ≡ number of incident particles, per unit time, per unit area

Per unit time, the number of particles entering an area $d\sigma$ is:

$$dN = \mathcal{L}\, d\sigma$$

Recall $d\sigma = D(\theta)\, d\Omega$

$$= \mathcal{L}\, D(\theta)\, d\Omega$$

↑ # of particles per unit time

$$\Rightarrow \quad D(\theta) = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega}$$

Q: What are the units of luminosity?
A: $cm^{-2} s^{-1}$

This can be a nice way to find the differential cross-section experimentally, because once we know the luminosity of our beam, all we have to do is count the number of events in our detector in some solid angle over some period of time. Then we can determine the event rate by calculating

$$dN = \frac{N}{t}$$

N ← number of events in detector

t ← time of measurement

↑ that is, dN, the number of events per unit time

then dividing by the solid angle of our detector and the luminosity of our beam.

Example 10.2

Let's go back to our hard sphere scattering from Example 10.1. Recall that we found

$$b = R\cos\left(\frac{\vartheta}{2}\right)$$

So that tells us that

$$\frac{db}{d\vartheta} = -\frac{R}{2}\sin\left(\frac{\vartheta}{2}\right)$$

From that, we can determine the differential cross-section

$$\begin{aligned} D(\vartheta) &= \frac{b}{\sin\vartheta}\left|\frac{db}{d\vartheta}\right| \\ &= R\cos\left(\frac{\vartheta}{2}\right)\cdot\frac{1}{\sin\vartheta}\cdot\left|-\frac{R}{2}\sin\left(\frac{\vartheta}{2}\right)\right| \\ &= \frac{R^2}{2}\cos\left(\frac{\vartheta}{2}\right)\sin\left(\frac{\vartheta}{2}\right)\frac{1}{\sin\vartheta} \qquad \sin 2\vartheta = 2\sin\vartheta\cos\vartheta \\ &= \frac{R^2}{2}\,\frac{\sin\vartheta}{2}\cdot\frac{1}{\sin\vartheta} \\ &= \frac{R^2}{4} \end{aligned}$$

Now we can integrate that to find the total cross-section

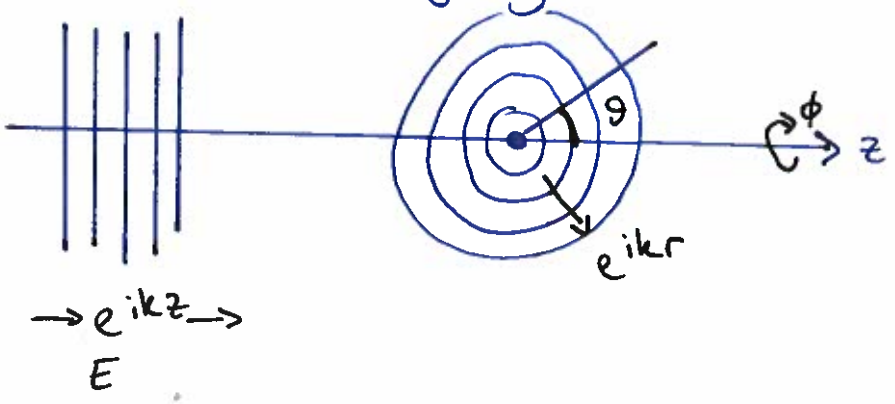
$$\begin{aligned} \sigma &= \int D(\vartheta)\,d\Omega \\ &= \int\frac{R^2}{4}\,d\Omega \\ &= \frac{R^2}{4}\int d\Omega \qquad \int d\Omega = \text{total solid angle} = 4\pi \\ &= \frac{R^2}{4}\cdot 4\pi = \pi R^2 \end{aligned}$$

Notice that in this case the total cross-section is just the area of the circle - that is, the area that the beam "sees" of the target

Quantum scattering

When we go from classical to quantum scattering, we move from a picture involving collisions of little balls to a picture involving wavefunctions. How does that change things?

Instead of a ball-like incident particle, we now have an incident plane-wave solution. When this plane wave encounters a scattering target (or "centre"), it scatters into an outgoing spherical wave



We assume the system is symmetric in ϕ, otherwise $f(\vartheta) \rightarrow f(\vartheta, \phi)$.

Thus, we expect our wavefunction to take the form

$$\psi(r,\vartheta) \simeq A\left[e^{ikz} + f(\vartheta)\,\frac{e^{ikr}}{r}\right]$$

at least for sufficiently large r.

The factor of $\frac{1}{r}$ ensures that the wavefunction decays sufficiently fast to conserve probability at large r.

Here the wavenumber is $k = \frac{\sqrt{2mE}}{\hbar}$

The unknown function $f(\theta)$ is the scattering amplitude

Basically the whole point of scattering theory is to determine this scattering amplitude, because it tells us the probability of scattering in a given direction θ.

In fact

$$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

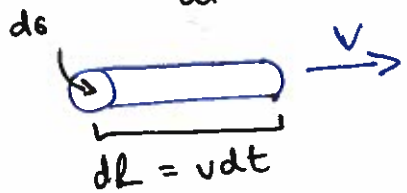
This follows from the fact that the probability that an incident particle (with speed v) passes through an infinitesimal area $d\sigma$ in time dt is

$$dP = |\psi_{incident}|^2 dV \quad \text{with } \psi_{incident} \simeq A e^{ikz}$$

$$= |A|^2 v\,dt\,d\sigma \quad (2)$$

and $dV = \underbrace{v\,dt}_{\text{length passing a point in time } dt}\, d\sigma$ ← cross-sectional area passing a point

This is equal to the probability that the "particle" scatters into the corresponding solid angle $d\Omega$



$$dP = |\psi_{scattered}|^2 dV \quad \text{with } \psi_{scattered} \simeq A\frac{e^{ikr}}{r} f(\theta)$$

$$= |A|^2 \frac{|f|^2}{r^2} (v\,dt) r^2 d\Omega \quad (1)$$

$(1) = (2) \Rightarrow |A|^2 v\,dt\,d\sigma = |A|^2 \frac{|f|^2}{r^2} v\,dt\,r^2 d\Omega \Rightarrow d\sigma = |f|^2 d\Omega$

↑ this leads to equation

Partial wave analysis

Chapter 10.2

The first of the techniques that we will use to solve for the scattering amplitude is partial wave analysis, in which we break the problem down into angular momentum components and solving those separately using the boundary conditions.

Recall that the Schrödinger equation for a spherically symmetric potential is separable

$$\psi(r,\theta,\phi) = R(r)\,Y_\ell^m(\theta,\phi)$$

↑ radial piece (p. 138) ↑ spherical harmonics (p. 137)

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(rR(r)) + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right] rR(r) = E\,rR(r)$$

↑ called "u" in the textbook

If r is enormous then $V(r)$ goes to zero and $\frac{1}{r}$ also vanishes so that the equation reduces to

$$\frac{d^2}{dr^2}(rR(r)) = -\frac{2mE}{\hbar^2}(rR(r)) \equiv -k^2(rR(r))$$

The solution to this is

$$rR(r) = Ce^{ik\cdot r} + De^{-ik\cdot r}$$

$Ce^{ik\cdot r}$: outgoing spherical wave

$De^{-ik\cdot r}$: incoming spherical wave

Since we want the incoming wave to be a plane wave, and the scattered wave is an outgoing spherical wave, we naturally have $D = 0$.

Thus, for very large r

$$R(r) \sim \frac{e^{ik\cdot r}}{r}$$

Q: Large compared to what?

we're assuming a localised scattering potential/target

If r is merely large, and not ginormous, then $V(r)$ can be ignored, but the $\frac{1}{r}$ term cannot, so our equation becomes

$$\frac{d^2}{dr^2}(rR(r)) - \frac{\ell(\ell+1)}{r^2}(rR(r)) = -k^2(rR(r))$$

see p. 140

which has general solution

$$rR(r) = Ar\, j_\ell(kr) + Br\, n_\ell(kr)$$

$j_\ell(kr)$: spherical Bessel function of order ℓ

$$j_\ell(x) \equiv (-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\sin x}{x}$$

$n_\ell(kr)$: spherical Neumann function of order ℓ

$$n_\ell(x) = -(-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\cos x}{x}$$

Unfortunately neither j_ℓ nor n_ℓ are incoming or outgoing waves. Thankfully we can construct linear combination that are - these are the Hankel functions

$$h_\ell^{(1)}(x) \equiv j_\ell(x) + i n_\ell(x)$$ ← Hankel function of the first kind

$$h_\ell^{(2)}(x) \equiv j_\ell(x) - i n_\ell(x)$$ ← Hankel function of the second kind

For large enough r,

$$h_\ell^{(1)}(x) \sim \frac{e^{ix}}{x} \quad \text{and} \quad h_\ell^{(2)}(x) \sim \frac{e^{-ix}}{x}$$

and again we want the outgoing spherical wave, so we need $h_\ell^{(1)}(kr)$. In other words

$$R(r) \sim h_\ell^{(1)}(kr)$$

that is, where $V(r) = 0$

In fact, the exact wavefunction in the exterior region is

$$\psi(r,\theta,\phi) = A\left\{e^{ikz} + \sum_{\ell,m} c_{\ell,m}\, h_\ell^{(1)}(kr)\, Y_\ell^m(\theta,\phi)\right\}$$

- $c_{\ell,m}$: expansion coefficients
- $h_\ell^{(1)}(kr)$: spherical Hankel functions of the first kind
- $Y_\ell^m(\theta,\phi)$: spherical harmonics

Now, we're not trying to solve the whole scattering problem in general, so we can assume that the potential is spherically symmetric (in fact, we already did, near the start of this whole thing). Thus the wavefunction can't depend on ϕ.

Since the spherical harmonics are the pieces that depend on ϕ, and they go like

$$Y_\ell^m(\theta,\phi) \sim e^{im\phi}$$

this tells us that only the $m=0$ contributions survive.

But we know (or we look up on page 137) that

$$Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

↑ ℓ^{th} Legendre polynomial

Finally we chose to write the expansion coefficients as

$$C_{\ell,0} \equiv i^{\ell+1} k \sqrt{4\pi(2\ell+1)}\, a_\ell$$

where a_ℓ is the ℓ^{th} partial wave amplitude

Thus we end up with

$$\boxed{\psi(r,\theta) = A\left\{e^{ikz} + k\sum_{\ell=0}^{\infty} i^{\ell+1}(2\ell+1)\, a_\ell\, h_\ell^{(1)}(kr)\, P_\ell(\cos\theta)\right\}}$$

For very large r we have

$$h_\ell^{(1)}(kr) \sim (-i)^{\ell+1}\frac{e^{ikr}}{kr}$$

$$\Rightarrow \psi(r,\theta) \simeq A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}$$

↑ $f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\, a_\ell\, P_\ell(\cos\theta)$

Thus, in order to calculate the scattering amplitude, "all" we need to do is calculate a bunch of partial wave amplitudes. Then, we plug our expression

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell\, P_\ell(\cos\theta)$$

into our known relations

$$D(\theta) = |f(\theta)|^2 = \sum_{\ell,\ell'} (2\ell+1)(2\ell'+1)\, a_\ell^* a_{\ell'} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

and

$$\begin{aligned} \sigma &= \int D(\theta)\, d\Omega \\ &= \int \sum_{\ell,\ell'} (2\ell+1)(2\ell'+1)\, a_\ell^* a_{\ell'} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d\Omega \\ &= \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \sum_{\ell,\ell'} (2\ell+1)(2\ell'+1)\, a_\ell^* a_{\ell'} \\ &\qquad \times P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \\ &= 2\pi \cdot \sum_{\ell,\ell'} (2\ell+1)(2\ell'+1)\, a_\ell^* a_{\ell'} \cdot \frac{2}{2\ell+1} \delta_{\ell\ell'} \\ &= 4\pi \sum_{\ell} \frac{(2\ell+1)^2}{(2\ell+1)} |a_\ell|^2 \\ &= 4\pi \sum_{\ell=0}^{\infty} (2\ell+1) |a_\ell|^2 \end{aligned}$$

$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$

The Legendre polynomials are orthonormal. See Problem 4.8.

to calculate the differential and total cross-sections respectively.

Clearly our job is to determine the partial wave amplitudes. But how do we do this?

First we solve the Schrödinger equation in the <u>interior region</u> (where the scattering potential is non zero) and then we match to the exterior region (where the scattering potential is zero – recall we have to assume the potential vanishes sufficiently quickly at large r for this to all work).

This is a bit tricky at the moment, however, since the incoming plane wave is expressed in terms of Cartesian coordinates (e^{ikz}). To make this more consistent, we use the fact that

$$\sum_{\ell,m} \left(A_{\ell,m}\, j_\ell(kr) + B_{\ell,m}\, n_\ell(kr) \right) Y_\ell^m(\theta,\phi)$$

is a solution to the Schrödinger equation for $V(r) = 0$ (which is the potential for which e^{ikz} satisfies the Schrödinger equation). So we must be able to rewrite e^{ikz} in this form. The Neumann function diverges at $r=0$, and there is no ϕ dependence in the variable z (we choose axes so $z = r\cos\theta$), which means only $m=0$ contributions are needed. We end up with

$$e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta)$$

↑ Rayleigh's formula

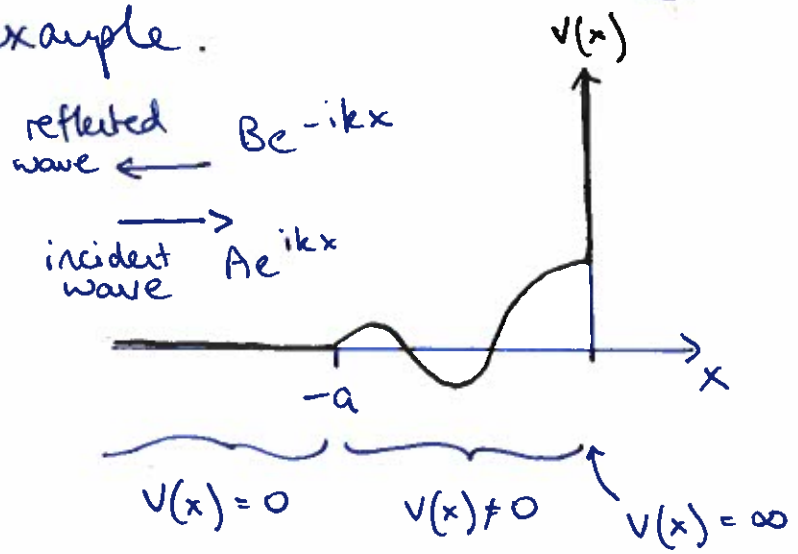
$$\Rightarrow \boxed{\psi(r,\theta) = A \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) \left[j_\ell(kr) + ika_\ell h_\ell^{(1)}(kr) \right] P_\ell(\cos\theta)}$$

Phase shifts

Chapter 10.3

Scattered waves often scatter with a different phase to the incoming wave. This factor is called, not too originally, the phase shift. In many cases, along with the "angle" of the wave, the phase is the only thing that can change, because the amplitude is constrained by conservation of probability.

To look at an example of how a phase shift arises, we'll consider a slightly unnatural looking one-dimensional example.



Here there is an incident wave, travelling in the region $-\infty < x < -a$, where $V(x) = 0$. This is a plane wave $\psi_i(x) = Ae^{ikx}$.

In the region $-a \leq x < 0$ there is some nonzero potential

Then there is an infinite wall at $x = 0$, so that the incident wave is entirely reflected. In the region $-\infty < x < -a$ this reflected wave is just a plane wave travelling in the opposite direction, $\psi_r(x) = Be^{-ikx}$.

The simplest case occurs if $V(x) = 0$ in the intermediate region $(-a < x < 0)$ as well. In that case, the wavefunction must vanish at $x = 0$ (where $V(x=0) = \infty$), &

$$\varphi_i(0) + \varphi_r(0) = A + B = 0 \quad \Rightarrow A = -B.$$

Thus the total wavefunction is

$$\varphi(x) = A\left(e^{ikx} - e^{-ikx}\right) \qquad x < 0,\ V(x) = 0 \text{ in } -a \le x < 0$$

But if $V(x) \neq 0$ in the region $-a \le x < 0$, then conservation of probability only requires that $|A|^2 = |B|^2$ in the plane-wave region, $x < -a$.

In that case, the general solution for $x < -a$ is

$$\varphi(x) = A\left(e^{ikx} - e^{i(2\delta - kx)}\right)$$

so that $B = -e^{2i\delta}A$ and $|B|^2 = |A|^2$.

Factor of 2 is just convention.

Then the problem comes down to calculating the phase shift as a function of k (and therefore E), given a specified potential in the intermediate region. We have used our physical knowledge (probability conservation) to reduce the problem from calculating a complex number (B, with two unknowns) to a single phase shift (δ, with one unknown number).

N.B. δ is real.

In three dimensions, we have a plane wave incident along the z direction ($\varphi_i(z) = Ae^{ikz}$), so it has no angular momentum in the z direction. ↖

But this wave does carry total angular momentum, since Rayleigh's formula includes all possible ℓ.

> You can think of this physically - when the displacement and momentum are parallel, their cross product must vanish in that direction - or mathematically - Rayleigh's formula has no contributions from $m \neq 0$
>
> $$Ae^{ikz} = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

Angular momentum is conserved, because we're assuming the potential is spherically symmetric, so this applies to each component (that is, each partial wave). Each partial wave scatters independently and during that process the total angular momentum ℓ is conserved.

If there is no potential at all, in the scattering region, then the wavefunction is just $\varphi(r,\theta) = Ae^{ikz}$. We can write this as ← using Rayleigh's formula, equation 10.28

$$\varphi(r,\theta) = Ae^{ikz} = A\sum_{\ell=0}^{\infty} \underbrace{i^{\ell}(2\ell+1) j_\ell(kr) P_\ell(\cos\theta)}_{\varphi^{(\ell)}(kr,\theta)} = A\sum_{\ell=0}^{\infty} \varphi^{(\ell)}(kr,\theta)$$

$\varphi^{(\ell)}(kr,\theta)$ ← "partial wave"

But we can express the spherical Bessel function $j_\ell(x)$ in terms of spherical Hankel functions by inverting equation 10.19

$$j_\ell(x) = \frac{1}{2}\left[h_\ell^{(1)}(x) + h_\ell^{(2)}(x)\right]$$

$h_\ell^{(1)}(x) = j_\ell(x) + i n_\ell(x)$ ↑

$h_\ell^{(2)}(x) = j_\ell(x) - i n_\ell(x)$

Table 10.1 lists the asymptotic forms for these spherical Hankel functions as

$$h_\ell^{(1)}(x) \to \frac{1}{x}(-i)^{\ell+1} e^{ix}$$
$$h_\ell^{(2)}(x) \to \frac{1}{x} i^{\ell+1} e^{-ix}$$

for very large x

This means

$$j_\ell(x) = \frac{1}{2}\left[h_\ell^{(1)}(x) + h_\ell^{(2)}(x)\right]$$
$$\sim \frac{1}{2}\left[\frac{1}{x}(-i)^{\ell+1} e^{ix} + \frac{1}{x} i^{\ell+1} e^{-ix}\right]$$
$$= \frac{1}{2x}\left[(-i)\cdot(-i)^\ell e^{ix} + i\cdot(i)^\ell e^{-ix}\right]$$
$$= \frac{1}{2ix}\left[(-i)^\ell e^{ix} - i^\ell e^{-ix}\right]$$
$$=$$

And plugging this into our expression gives

$$\psi(r,\theta) \simeq A\sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\left\{\frac{1}{2ikr}\left[(-i)^\ell e^{ikr} - i^\ell e^{-ikr}\right]\right\} P_\ell(\cos\theta)$$
$$= \frac{A}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left[i^\ell(-i)^\ell e^{ikr} - i^\ell\cdot i^\ell e^{-ikr}\right] P_\ell(\cos\theta)$$
$$= \frac{A}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{ikr} - (-1)^\ell e^{-ikr}\right] P_\ell(\cos\theta)$$

$i^\ell(-i)^\ell = (i\cdot -i)^\ell = 1^\ell = 1$

and $i^\ell\cdot i^\ell = (i^2)^\ell = (-1)^\ell$

outgoing spherical wave

incoming spherical wave - from the incident plane wave

The incoming wave is unaffected by the scattering potential, but the scattered outgoing wave picks up a phase shift, just as it did for the one-dimensional case.

So, including the phase shift, we have

$$\psi(r,\theta) = \frac{A}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\underbrace{\left[e^{i(kr+2\delta_\ell)} - (-1)^\ell e^{-ikr}\right]P_\ell(\cos\theta)}_{\psi^{(\ell)}(r,\theta)}$$

for large r!

Note each partial wave picks up a different phase shift, because they scatter independently.

In this three-dimensional case, we have an incoming spherical wave (which comes from the $h_\ell^{(2)}$ component in e^{ikz}) that is phase shifted by an amount δ_ℓ on the way in and again by δ_ℓ on the way out (hence the "$2\delta_\ell$"), leading to an outgoing, phase-shifted spherical wave (which comes partly from $h_\ell^{(1)}$ in e^{ikz} and partly from the scattered wave itself).

↑ labelled by ℓ

So what have we done here? We've expressed our total wavefunctions in terms of partial waves, which incorporate a phase shift. In the previous section, we expressed our total wavefunction in terms of partial wave amplitudes.

By comparing these expressions, we can relate them, at least at large r.

We first write the total wavefunction in terms of the partial wave amplitudes

partial wave amplitudes

$$\psi(r,\theta) = A\sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)\left[j_\ell(kr) + ika_\ell h_\ell^{(1)}(kr)\right]P_\ell(\cos\theta)$$

↑ Equation 10.29

Equation 10.42 ↓

$$\simeq A\sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)\left\{\frac{1}{2ikr}\left[(-i)^{\ell+1}e^{ikr} + i^{\ell+1}e^{-ikr}\right]\right.$$

Table 10.1 ↓

$$\left. + ika_\ell\left[\frac{1}{kr}(-i)^{\ell+1}e^{ikr}\right]\right\}P_\ell(\cos\theta)$$

Use the same tricks as page 142 of the notes ↓

$$= \frac{A}{r}\sum_{\ell=0}^{\infty}(2\ell+1)\left\{\frac{1}{2ik}\left[e^{ikr} - (-1)^{\ell}e^{-ikr}\right] + a_\ell e^{ikr}\right\}P_\ell(\cos\theta) \qquad (*)$$

Now we can compare that to

$$\psi(r,\theta) \simeq \frac{A}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{i(kr+2\delta_\ell)} - (-1)^{\ell}e^{-ikr}\right]P_\ell(\cos\theta) \qquad (**)$$

To relate a_ℓ and δ_ℓ, we can write

$$(*) - (**) = 0$$

or

$$\frac{A}{r}\sum_{\ell=0}^{\infty}(2\ell+1)\left\{\frac{1}{2ik}\left[e^{ikr} - (-1)^{\ell}e^{-ikr}\right] + a_\ell e^{ikr}\right\}P_\ell(\cos\theta)$$

$$- \frac{A}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{i(kr+2\delta_\ell)} - (-1)^{\ell}e^{-ikr}\right]P_\ell(\cos\theta) = 0$$

$$\Rightarrow \frac{A}{r}\sum_{\ell=0}^{\infty}(2\ell+1)\left\{\frac{1}{2ik}\left[e^{ikr} - (-1)^{\ell}e^{-ikr}\right] + a_\ell e^{ikr}\right.$$

$$\left. - \frac{1}{2ik}\left[e^{i(kr+2\delta_\ell)} - (-1)^{\ell}e^{-ikr}\right]\right\}P_\ell(\cos\theta) = 0$$

Each term in the sum is independent, so each must satisfy

$$\frac{1}{2ik}\left[e^{ikr} - (-1)^\ell e^{-ikr}\right] + a_\ell e^{ikr} - \frac{1}{2ik}\left[e^{i(kr+2\delta_\ell)} - (-1)^\ell e^{-ikr}\right] = 0$$

or

$$a_\ell e^{ikr} = \frac{1}{2ik} e^{i(kr+2\delta_\ell)} - \frac{1}{2ik} e^{ikr}$$

$$\Rightarrow a_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) = \frac{e^{i\delta_\ell}}{k}\sin(\delta_\ell)$$

This is what we're looking for! A relationship between the phase shift, δ_ℓ, and the partial wave amplitude, a_ℓ.

Then we can plug this into

← Equation 10.25

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell\, P_\ell(\cos\theta)$$

↑ scattering amplitude

$$= \sum_{\ell=0}^{\infty} (2\ell+1)\, \frac{e^{i\delta_\ell}}{k} \sin(\delta_\ell)\, P_\ell(\cos\theta)$$

and from that

← Equation 10.27

$$\sigma = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1)\, |a_\ell|^2$$

↑ total cross-section

$$= \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2(\delta_\ell)$$

So why do all this? Basically, it makes everything easier! We've used angular momentum conservation to write everything in terms of real numbers (the phase shifts) not complex numbers (the a_ℓ

↑ one real parameter ↑ complex

Born approximations

Chapter 10.4

Partial wave amplitudes and phase shifts provide one method for analysing scattering theory/problems. Before we investigate a second method – the Born approximation.

Obviously this movie was less successful than the Born Identity.

For that, we need the integral representation of the Schrödinger equation

$$\psi(\bar{r}) = \psi_0(\bar{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\bar{r}-\bar{r}_0|}}{|\bar{r}-\bar{r}_0|} V(\bar{r}_0)\,\psi(\bar{r}_0)\,d^3\bar{r}_0$$

$\psi_0(\bar{r})$: solution to free-particle Schrödinger equation $\nabla^2\psi_0 = -k^2\psi_0$ (i.e. $V(\bar{r}) = 0$)

$k = \frac{\sqrt{2mE}}{\hbar}$

This looks like a solution to the Schrödinger equation, for any potential, but it is not! The integral requires $\psi(\bar{r}_0)$, so we have to know the solution itself, before we can carry out the integral (note it is not $\psi_0(\bar{r}_0)$ that appears in the integral).

The integral representation is entirely equivalent to the usual (differential) form that we're all used to.

To derive this expression, we need to use the method of Green's functions.

There is quite a lot of debate, a surprising amount, really, about "Green" vs "Green's".

A Green's function of a linear differential operator is the solution to the equation

$$\mathcal{L}\, G(x,y) = \delta(x-y)$$

← Dirac delta function "source"

$\mathcal{L}$: differential operator (such as ∇^2)

$G(x,y)$: ← Green's function – note I use two arguments for this function, the textbook has only one, which is less general

The Green's function represents the response of a system (roughly speaking, the part defined by the linear differential operator) to an impulse (basically a delta function spike). The Green's function is useful because it allows us to solve the more general inhomogeneous differential equation

$$\mathcal{L}\, u(x) = f(x)$$

$u(x)$: general solution

$f(x)$: some general "source" function

Starting from the definition of the Green's function, we have

$$\int [\mathcal{L}\, G(x,y)]\, f(y)\, dy = \int [\delta(x-y)]\, f(y)\, dy$$
$$= f(x)$$

But $\mathcal{L}$ is linear, and acts only on x (not y), so the left hand side can be written as

$$\int [\mathcal{L}\, G(x,y)]\, f(y)\, dy = \mathcal{L} \underbrace{\int G(x,y)\, f(y)\, dy}_{u(x)} = f(x)$$

← from right hand side above

Thus, if we can find the Green's function of the operator $\mathcal{L}$, then the general solution can be constructed from

$$u(x) = \int G(x,y) f(y)\, dy.$$

This is all very nice, but how does this help us with the Schrödinger equation? We start with the usual (time-independent) form

$$-\frac{\hbar^2}{2m}\nabla^2\varphi + V\varphi = E\varphi$$

$$\Rightarrow \quad \nabla^2\varphi + \frac{2mE}{\hbar^2}\varphi = \frac{2m}{\hbar^2}V\varphi$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$Q = \frac{2m}{\hbar^2}V\varphi$$

$$(\nabla^2 + k^2)\varphi = Q$$

This is an inhomogeneous linear differential equation, with

$$\mathcal{L} = \nabla^2 + k^2$$

$$u(x) = \varphi(x)$$

$$f(x) = Q(x) = \frac{2m}{\hbar^2}V(x)\varphi(x)$$

So if we can solve for the Green's function of the differential operator, $\mathcal{L}$, we can find a solution to the whole equation!

Except. Our source contains $\varphi(x)$ itself (as we've already seen), so the best we'll get is an integral representation of the equation, rather than a solution.

Clearly we first need to find the Green's function, which satisfies

$$\hat{L}\, G(\bar{r}, \bar{r}_0) = \delta(\bar{r} - \bar{r}_0)$$

or

$$(\nabla^2 + k^2)\, G(\bar{r}, \bar{r}_0) = \delta(\bar{r} - \bar{r}_0)$$

The easiest way to do this is by Fourier transform.

We start by writing the delta function as

$$\delta^3(\bar{r}) = \frac{1}{(2\pi)^3} \int e^{i\bar{q}\cdot\bar{r}}\, d^3\bar{q}$$

This is very much a physicist's "definition" of the delta function. See problem 2.26 for a small discussion of this. Whatever you think of it, it works for our purposes

Then we write the Green's function as

$$G(\bar{r}, \bar{r}_0) = \frac{1}{(2\pi)^3} \int e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)} \tilde{G}(\bar{q})\, d^3\bar{q}$$

slightly different normalisation to the textbook

unknown function - to be determined!

This gives us

$$(\nabla^2 + k^2)\, G(\bar{r}, \bar{r}_0) = (\nabla^2 + k^2) \left\{ \frac{1}{(2\pi)^3} \int e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)} \tilde{G}(\bar{q})\, d^3\bar{q} \right\}$$

$$= \frac{1}{(2\pi)^3} \int (\nabla^2 + k^2) e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)} \tilde{G}(\bar{q})\, d^3\bar{q} \qquad \nabla^2 e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)} = -|\bar{q}|^2 e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)}$$

$$= \frac{1}{(2\pi)^3} \int (-|\bar{q}|^2 + k^2) e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)} \tilde{G}(\bar{q})\, d^3\bar{q}$$

Now we equate that to the integral representation of the delta function

$$(\nabla^2 + k^2)\, G(\bar{r}, \bar{r}_0) = \delta(\bar{r} - \bar{r}_0)$$

$$\Rightarrow \frac{1}{(2\pi)^3} \int (-|\bar{q}|^2 + k^2)\, e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)}\, \tilde{G}(\bar{q})\, d^3\bar{q} = \frac{1}{(2\pi)^3} \int e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)}\, d^3\bar{q}$$

$$\Rightarrow \frac{1}{(2\pi)^3} \int \left\{ (-|\bar{q}|^2 + k^2)\, \tilde{G}(\bar{q}) - 1 \right\} e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)}\, d^3\bar{q} = 0$$

$$\Rightarrow (-|\bar{q}|^2 + k^2)\, \tilde{G}(\bar{q}) - 1 = 0 \qquad \text{let } q^2 = |\bar{q}|^2$$

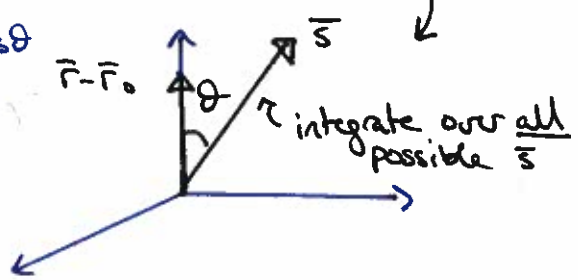
$$\Rightarrow \tilde{G}(\bar{q}) = \frac{1}{k^2 - q^2}$$

We can plug this into our integral expression for the Green's function to obtain

$$G(\bar{r}, \bar{r}_0) = \frac{1}{(2\pi)^3} \int e^{i\bar{q}\cdot(\bar{r}-\bar{r}_0)}\, \frac{1}{k^2 - q^2}\, d^3\bar{q}$$

Now we "just" need to do the integration. First we notice that $(\bar{r} - \bar{r}_0)$ is fixed and independent of $\bar{q}$. So we can choose our integration coordinates such that $\bar{q}\cdot(\bar{r} - \bar{r}_0) = q|\bar{r} - \bar{r}_0|\cos\vartheta$

$$\Rightarrow \int \frac{e^{i\bar{q}\cdot|\bar{r}-\bar{r}_0|}}{k^2 - q^2}\, d^3\bar{q} = \int_0^{2\pi} d\phi \int_0^{\pi} d\vartheta \sin\vartheta \int_0^{\infty} q^2 dq\, \frac{e^{iq|\bar{r}-\bar{r}_0|\cos\vartheta}}{k^2 - q^2}$$



$$= 2\pi \cdot \int_0^{\infty} dq\, \frac{q^2}{k^2 - q^2} \int_0^{\pi} d\vartheta\, e^{iq|\bar{r}-\bar{r}_0|\cos\vartheta} \sin\vartheta$$

ϕ integration

$$\int_0^{\pi} d\vartheta \sin\vartheta\, e^{iq|\bar{r}-\bar{r}_0|\cos\vartheta} = \frac{-e^{iq|\bar{r}-\bar{r}_0|\cos\vartheta}}{iq|\bar{r}-\bar{r}_0|}\Bigg|_0^{\pi} = \frac{2\sin(q|\bar{r}-\bar{r}_0|)}{q|\bar{r}-\bar{r}_0|}$$

So our Green's function is

$$G(\bar{r},\bar{r}_0) = \frac{1}{(2\pi)^3}\cdot 2\pi\cdot\frac{2}{|\bar{r}-\bar{r}_0|}\int_0^\infty \frac{\sin(q|\bar{r}-\bar{r}_0|)}{q(k^2-q^2)}\,q^2\,dq$$

$$= \frac{1}{2\pi^2|\bar{r}-\bar{r}_0|}\int_0^\infty \frac{\sin(q|\bar{r}-\bar{r}_0|)}{k^2-q^2}\,q\,dq$$

↓ integral is even under $q\leftrightarrow -q$, so integral from 0 to ∞ is half integral from $-\infty$ to ∞.

$$= \frac{1}{4\pi^2|\bar{r}-\bar{r}_0|}\int_{-\infty}^\infty \frac{\sin(q|\bar{r}-\bar{r}_0|)}{k^2-q^2}\,q\,dq$$

We will use contour integration to do this integral, but first we write it as

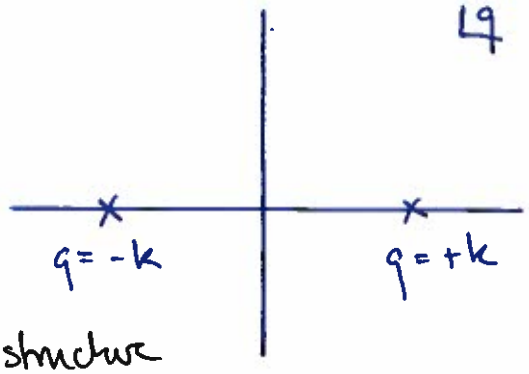
↑ and the Cauchy integral formula and residue theorem

$$G(\bar{r},\bar{r}_0) = \frac{1}{4\pi^2|\bar{r}-\bar{r}_0|}\int_{-\infty}^\infty \frac{q}{2i}\left(\frac{e^{iq|\bar{r}-\bar{r}_0|}-e^{-iq|\bar{r}-\bar{r}_0|}}{(k-q)(k+q)}\right)dq$$

$$= \frac{1}{8\pi i|\bar{r}-\bar{r}_0|}\left[\underbrace{\int_{-\infty}^\infty q\frac{e^{iq|\bar{r}-\bar{r}_0|}}{(k-q)(k+q)}\,dq}_{I_1} - \underbrace{\int_{-\infty}^\infty q\frac{e^{iq|\bar{r}-\bar{r}_0|}}{(k-q)(k+q)}\,dq}_{I_2}\right]$$

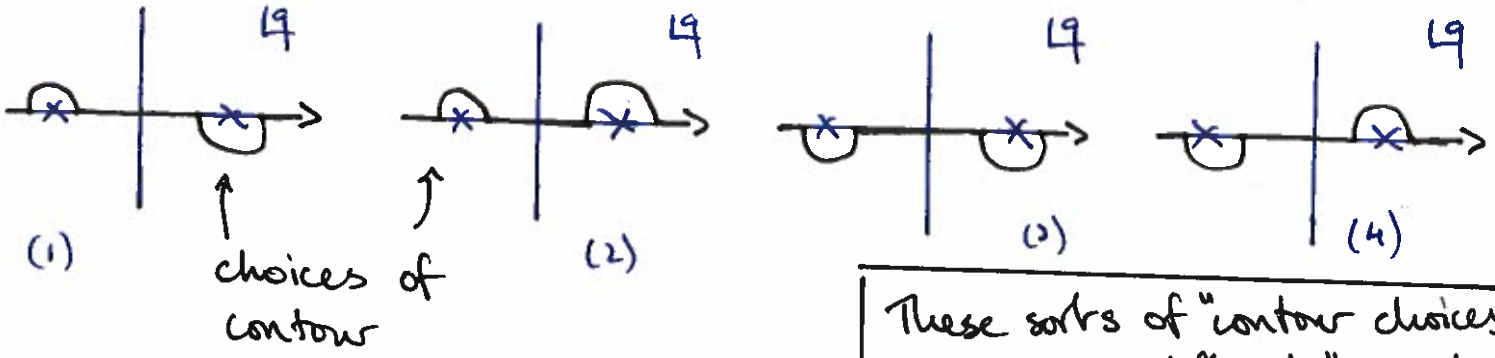
Both I_1 and I_2 have two poles, at $q=\pm k$. Our integration contour runs along the real axis, and whether we close that contour in the upper or lower half-plane is determined by the exponentials in the numerator.

Even before we get to closing the contour, we have to figure out what to do with the contour on the real axis, which runs straight through both poles.



pole structure in the complex q-plane

There are four obvious choices ← There are really an infinite number, but the rest are kinda weird.



These sorts of "contour choices" give rise to different "propagators" in quantum field theory.

When we choose one of the contour prescriptions, we then have to close the contour in one of the half-planes.

For I_1, we want to close in the upper half plane, because

$$e^{iq|\bar{r}-\bar{r}_0|} \to e^{-|q||\bar{r}-\bar{r}_0|}$$

when q is purely imaginary and positive, that is, $q \to i|q|$

For I_2, we want to close in the lower half plane, because

$$e^{-iq|\bar{r}-\bar{r}_0|} \to e^{-|q||\bar{r}-\bar{r}_0|}$$

when q is purely imaginary and negative, that is, $q \to -i|q|$

In both cases, we want the contour at infinity to give vanishing contribution, which dictates our choice.

If we follow the textbook and choose contour 1 above, then closing in the upper half plane for I_1 encloses the $q=+k$ pole and we obtain

$$I_1 = \oint \frac{q e^{iq|\bar{r}-\bar{r}_0|}}{(q+k)(k-q)}\,dq = -2\pi i \left(\frac{q e^{iq|\bar{r}-\bar{r}_0|}}{q+k}\right)_{q=k} = -i\pi e^{ik|\bar{r}-\bar{r}_0|}$$

For I_2, we close in the lower half plane and pick up the pole at $q = -k$. We also pick up a minus sign because we traverse the contour in a clockwise direction

$$I_2 = -\oint \frac{q\, e^{iq|\bar{r}-\bar{r}_0|}}{(k-q)(k+q)}\, dq = 2\pi i \left(\frac{q\, e^{-iq|\bar{r}-\bar{r}_0|}}{q-k} \right)_{q=-k} = +i\pi e^{ik|\bar{r}-\bar{r}_0|}$$

Let's plug those back into our expression for the Green's function

$$G(\bar{r},\bar{r}_0) = \frac{1}{8\pi^2 i\,|\bar{r}-\bar{r}_0|}\left[I_1 - I_2\right]$$

$$= \frac{1}{8\pi^2 i\,|\bar{r}-\bar{r}_0|}\left[-i\pi e^{ik|\bar{r}-\bar{r}_0|} - i\pi e^{ik|\bar{r}-\bar{r}_0|}\right]$$

$$= -\frac{e^{ik|\bar{r}-\bar{r}_0|}}{4\pi\,|\bar{r}-\bar{r}_0|}$$

<u>This</u> is the Green's function for

$$(\nabla^2 + k^2)\, G(\bar{r}-\bar{r}_0) = \delta^{(3)}(\bar{r}-\bar{r}_0)$$

We can plug it into our solution for the Schrödinger equation

$$(\nabla^2 + k^2)\, \psi_Q(\bar{r}) = Q(\bar{r})$$

to obtain

$$\psi_Q(\bar{r}) = \int G(\bar{r}-\bar{r}_0)\, Q(\bar{r}_0)\, d^3\bar{r}_0$$

$$= -\int \frac{e^{ik|\bar{r}-\bar{r}_0|}}{4\pi\,|\bar{r}-\bar{r}_0|} \cdot \frac{2m}{\hbar^2} V(\bar{r}_0)\, \psi(\bar{r}_0)\, d^3\bar{r}_0$$

But we're not quite there yet! There is an ambiguity in the Green's function that corresponds to our choice of how to navigate the poles on the real axis. If $G(\bar{r}, \bar{r}_0)$ is a solution to $(\nabla^2 + k^2) G(\bar{r}, \bar{r}_0) = \delta^{(3)}(\bar{r} - \bar{r}_0)$, then so is $G(\bar{r}, \bar{r}_0) + G_0(\bar{r})$, where $G_0(\bar{r})$ satisfies the homogeneous equation

$$(\nabla^2 + k^2) G_0(\bar{r}) = 0$$

Thus the most general solution to the Schrödinger equation is

$$\psi(\bar{r}) = \psi_0(\bar{r}) + \psi_Q(\bar{r})$$
$$= \psi_0(\bar{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\bar{r} - \bar{r}_0|}}{|\bar{r} - \bar{r}_0|} V(\bar{r}_0) \psi(\bar{r}_0) \, d^3\bar{r}_0$$

← what we started this section with!

As we said earlier, this is actually an integral representation of the Schrödinger equation, <u>not</u> a solution, since $\psi(\bar{r})$ appears in the integral. It is, however, completely equivalent to the differential form we're used to.

<u>Some notes</u>:

- This derivation is pretty dodgy. It relies on an ill-defined integral representation of the delta function and an integral representation of the Green's function that is also ill-defined, because, strictly speaking, the Fourier transform does not exist. These issues can be fixed, but it requires more formal mathematics than we could possibly want to use.

- Different deformations of the real axis contour lead to different Green's functions, although they are all solutions to the same equation. This is related to the non-existence of the integral and, practically speaking, it doesn't matter which one you choose. Green's functions are always families of solutions (up until you impose boundary conditions).

First Born approximation.

So now we have an integral representation of the Schrödinger equation, what will we do with it?

Well, first of all, we'll make some approximations (but not the Born approximation, not yet). First we'll assume that the scattering potential is localised around some point $\bar{r}_0$, so that the potential drops to zero outside some finite region, and that we're also only considering the asymptotic behaviour of the wavefunction at large distances.

This makes sense because we're considering the quantum analogue of bouncing a metal ball off a bowling ball, which we don't normally think of as infinite in size.

Thus we take

$$|\bar{r}| \gg |\bar{r}_0|$$

$$\Rightarrow |\bar{r} - \bar{r}_0|^2 = r^2 + r_0^2 - 2\bar{r}\cdot\bar{r}_0$$

$$\simeq r^2\left(1 - 2\frac{\bar{r}\cdot\bar{r}_0}{r^2}\right)$$

$$\Rightarrow |\bar{r} - \bar{r}_0| \simeq r - \hat{\underline{r}}\cdot\bar{r}_0$$

$r = \sqrt{r^2} = \sqrt{|\bar{r}|^2}$

$\hat{r} = \bar{r}/r$

Now we introduce the vector $\bar{k}$, which is

$$\bar{k} = k\hat{r}$$

So the exponential factor is now

$$e^{ik|\bar{r}-\bar{r}_0|} \simeq e^{ik(r-\hat{r}\cdot\bar{r}_0)}$$

$$= e^{ikr}e^{-i\bar{k}\cdot\bar{r}_0}$$

and

$$\frac{e^{ik|\bar{r}-\bar{r}_0|}}{|\bar{r}-\bar{r}_0|} \simeq \frac{e^{ikr}e^{-i\bar{k}\cdot\bar{r}_0}}{r-\hat{r}\cdot\bar{r}_0}$$

↓ the extra term in the denominator only contributes at higher order in r_0/r

$$\simeq \frac{e^{ikr}e^{-i\bar{k}\cdot\bar{r}_0}}{r}$$

Recall that our Schrödinger equation is

$$\psi(\bar{r}) = \psi_0(\bar{r}) - \frac{M}{2\pi\hbar^2}\int \frac{e^{ik|\bar{r}-\bar{r}_0|}}{|\bar{r}-\bar{r}_0|} V(\bar{r}_0)\psi(\bar{r}_0)\,d^3\bar{r}_0$$

and $\psi_0(\bar{r})$ is the solution of the "free" Schrödinger equation, which, for scattering problems is just an incident plane wave (travelling in the region in which $V=0$), so

$$\psi_0(\bar{r}) = Ae^{ikz}$$

Adding in our approximations above, we end up with

$$\psi(\bar{r}) \simeq Ae^{ikz} - \frac{M}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\bar{k}\cdot\bar{r}_0} V(\bar{r}_0)\psi(\bar{r}_0)\,d^3\bar{r}_0$$

At this point, we think back to our definition of the scattering amplitude, which was defined through

$$\varphi(r,\theta) \simeq A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\} \qquad \text{for large } r$$

Comparing these forms, we see $f(\theta)$ is

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\bar{k}\cdot\bar{r}_0}\, V(\bar{r}_0)\,\varphi(\bar{r}_0)\, d^3\bar{r}_0$$

This is an exact expression for the scattering amplitude. We are finally in a position to apply the <u>Born approximation</u>: we assume that the potential is weak enough in the scattering region that it barely affects the incoming plane wave. Thus

$$\varphi(\bar{r}_0) \simeq \varphi_0(\bar{r}_0)$$
$$= A e^{ikz_0} \qquad \downarrow\ \bar{k}' \equiv k\hat{z} \quad \leftarrow \bar{k}' \text{ is a vector with magnitude } k = \frac{\sqrt{2mE}}{\hbar}, \text{ in the direction of the incident wave.}$$
$$= A e^{i\bar{k}'\cdot\bar{r}_0}$$

This approximation is exact if $V(\bar{r}_0) = 0$.

With this (first) Born approximation, we find

$$f(\theta,\phi) \simeq -\frac{m}{2\pi\hbar^2 A}\int e^{-i\bar{k}\cdot\bar{r}_0}\, V(\bar{r}_0)\, A e^{i\bar{k}'\cdot\bar{r}_0}\, d^3\bar{r}_0$$

$\bar{k}$ has the same magnitude, but points in the scattered direction

or

$$\boxed{f(\theta,\phi) \simeq -\frac{m}{2\pi\hbar^2}\int e^{i(\bar{k}'-\bar{k})\cdot\bar{r}_0}\, V(\bar{r}_0)\, d^3\bar{r}_0}$$

$\bar{k}'-\bar{k}$ is the wavenumber transfer, so that $\hbar(\bar{k}-\bar{k}')$ is the <u>momentum transfer</u> of the process

There are two simplifications we can make:

1. low energy

In this case $k = |\bar{k}| = \frac{\sqrt{2mE}}{\hbar} \ll \frac{1}{r_0}$

Then the exponential factor is essentially constant in the region that $V(\bar{r}_0)$ is nonzero.

This gives us

$$f(\theta,\phi) \simeq -\frac{m}{2\pi\hbar^2}\int V(\bar{r}_0)\, d^3\bar{r}_0$$

this is the long wavelength limit, because the wavelength $\sim \frac{1}{k}$ and this is $\frac{1}{k} \gg r_0$, which characterises the size of the interaction region.

2. Spherically symmetric

In this case $V(\bar{r}) = V(r)$.

Then, if we define $\bar{K} = \bar{k}' - \bar{k}$, we can choose our axes so that

$$(\bar{k}' - \bar{k})\cdot\bar{r}_0 = |\bar{k}' - \bar{k}|\,|\bar{r}_0|\cos\theta_0 = Kr_0\cos\theta_0$$

and our integral becomes

$$f(\theta) \simeq -\frac{m}{2\pi\hbar^2}\int e^{iKr_0\cos\theta_0}\, V(r_0)\, r_0^2\, dr_0 \sin\theta_0\, d\theta_0\, d\phi_0$$

$$f(\theta) \simeq -\frac{2m}{\hbar^2 K}\int_0^\infty r_0\, V(r_0)\sin(Kr_0)\, dr_0$$

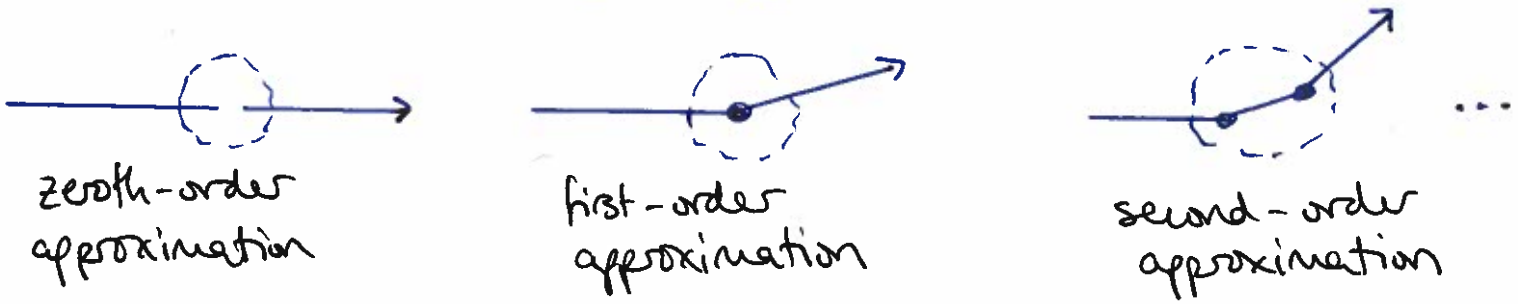
$\int_0^{2\pi} d\phi_0 = 2\pi$

$\int_0^{\pi} e^{iKr_0\cos\theta}\sin\theta_0\, d\theta_0 = \frac{2\sin(Kr_0)}{Kr_0}$ [see equation 10.59]

N.B. $K = |\bar{K}| = 2k\sin\frac{\theta}{2}$

We've only studied the first Born approximation. This is the first correction to the case of no scattering (when $V = 0$), which we can think of as the zeroth-order Born approximation.

Thus we can think of a series of Born approximations, which, given enough terms, would converge to the exact result for arbitrarily strong potentials.



We imagine that, within the scattering region, the incident wave experiences no, one, two, ... "kicks" (or impulses) from the scattering potential

To make this a little more formal, we start (again) from the Schrödinger equation

$$\psi(\bar{r}) = \psi_0(\bar{r}) - \frac{M}{2\pi\hbar^2}\int \frac{e^{ik|\bar{r}-\bar{r}_0|}}{|\bar{r}-\bar{r}_0|} V(\bar{r}_0)\,\psi(\bar{r}_0)\,d^3\bar{r}_0$$

Zeroth order approximation: $V(\bar{r}_0) = 0$

$$\Rightarrow \psi^{(0)}(\bar{r}) = \psi_0(\bar{r})$$

↑ recall this is a plane wave

First order approximation: $V(\bar{r}_0)$ is weak and $\varphi(\bar{r}_0) \approx \varphi^{(0)}(\bar{r}_0)$

$$\Rightarrow \varphi^{(1)}(\bar{r}) = \varphi_0(\bar{r}) - \frac{M}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\bar{k}\cdot\bar{r}_0} V(\bar{r}_0)\,\varphi^{(0)}(\bar{r}_0)\,d^3\bar{r}_0$$

↑ recall $\varphi^{(0)} = \varphi_0$

The second order approximation comes from approximating $\varphi(\bar{r}_0)$ not by $\varphi^{(0)}(\bar{r}_0) = \varphi_0(\bar{r}_0)$, but by $\varphi^{(1)}(\bar{r}_0)$

$$\Rightarrow \varphi^{(2)}(\bar{r}) = \varphi_0(\bar{r}) - \frac{M}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\bar{k}\cdot\bar{r}_0} V(\bar{r}_0)\,\varphi^{(1)}(\bar{r}_0)\,d^3\bar{r}_0$$

We can keep up this iterative process, improving our approximation with each step.

The physical picture we have is of a set of successive "kicks" within the scattering region, which we can sum up to give the full solution.

Apparently this was the inspiration for the path integral formalism, speaking of which...

And that's it for scattering in this course!