Learning outcomes

You will be able to:

- identify key problems encountered in formulating a consistent relativistic quantum mechanics
- write down the Dirac equation
- define the Clifford algebra
- distinguish spinors and vectors
- derive the Klein-Gordon equation from the Dirac equation
- solve the Dirac equation for a free particle at rest

Relativistic quantum mechanics

This course has been 100% entirely focussed on nonrelativistic quantum mechanics, in which the basic aim of the subject is to solve the (nonrelativistic) Schrödinger equation for a given potential. One of the reasons for this is that it's easier to learn one thing at a time – quantum mechanics is hard enough conceptually without the headache of special relativity on top of that – but the main reason, as we will see, is that trying to make sense of relativistic quantum systems will lead to some serious inconsistencies.

Ultimately these inconsistencies are solved by introducing quantum field theory (usually via the occupation number representation and the so-called "second quantisation", a name that I strongly dislike). The fundamental challenge for relativistic quantum mechanics is the reconciliation of the uncertainty principle and the mass-energy relation $E = mc^2$, which leads to particle nonconservation – a challenge that can be handled in the context of quantum field theory.

Now, quantum field theory is definitely awesome, but sadly we don't have a spare year to learn it right now.

check out my course in the autumn!

We, can, however, see why we <u>need</u> quantum field theory, by trying to generalise the Schrödinger equation to include relativistic particles. The result, the Dirac equation, was picked as "the most beautiful equation" in a poll by the BBC (so that settles that), brought together the two pillars of 20th Century particle physics, led to the successes of quantum field theory, and predicted the existence of antimatter. Which is pretty good.

Brief reminder on relativity

Before we get started on relativistic quantum mechanics, let's remind ourselves about special relativity.

Spacetime coordinates are represented by four-vectors

$$x^\mu = (x^0, x^1, x^2, x^3)$$

with scalar product

$$x_\mu x^\mu = \eta_{\mu\nu} x^\nu x^\mu$$

$$= x^0x^0 - x^1x^1 - x^2x^2 - x^3x^3$$

Einstein summation convention

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This is the spacetime "metric"

This scalar product is invariant under Lorentz transformations such as boosts (by v, in the z direction in this case)

$$x'^{\mu} = \begin{pmatrix} \gamma & 0 & 0 & -v/c^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v & 0 & 0 & \gamma \end{pmatrix} x^{\mu} \qquad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

The four velocity is

$$u^{\mu} = \gamma(c, \bar{u})$$

↑ 3-vector $\bar{u} = (u^x, u^y, u^z)$

$$= \frac{dx^{\mu}}{d\tau}$$

↖ proper time τ

$$\frac{dt}{d\tau} = \gamma$$

$$\begin{aligned} u_{\mu} u^{\mu} &= u^{\mu} u^{\nu} \eta_{\mu\nu} \\ &= u \cdot u = u^0 u^0 - u^1 u^1 - u^2 u^2 - u^3 u^3 \\ &= u^0 u_0 + u^1 u_1 + u^2 u_2 + u^3 u_3 \\ &= u^{\mu} u_{\mu} \end{aligned}$$

and the four velocity satisfies $u_{\mu} u^{\mu} = c^2$.

> This section uses Einstein summation convention:
> - repeated upstairs and downstairs indices are summed over
> - no more than two of the same indices
> - repeated indices must be up-down pairs

The four momentum is

$$p^{\mu} = m u^{\mu}$$
$$= \left(\frac{E}{c}, \bar{p} \right),$$

which satisfies $p_{\mu} p^{\mu} = m^2 c^2$. This leads to

$$E^2 = \bar{p}^2 c^2 + m^2 c^4.$$

We will "derive" the Dirac equation from this <u>dispersion relation</u>, with the momentum operator

> Relationship between velocity/momentum and energy.

$$p^{\mu} = i\hbar \frac{\partial}{\partial x_{\mu}}$$

Dirac equation

Starting from

$$E^2 - \vec{p}^2 c^2 = m^2 c^4$$

$$\left((\hat{p}^0)^2 - c^2 (\hat{\vec{p}})^2 - m^2 c^4 \right) \psi(x^\mu) = 0$$

we can use our operator prescription to write this as

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(x^\mu) + \hbar^2 c^2 \frac{\partial^2}{\partial x_i^2} \psi(x^\mu) = m^2 c^4 \psi(x^\mu)$$

the wavefunction is now a function of spacetime

this is the three-dimensional Laplacian

Schrödinger wrote this down, but discarded it, before realising the nonrelativistic limit is useful!

This is the relativistic version of the free Schrödinger equation - the Klein-Gordon equation. It turns out that this is the relativistic equation for a spin zero particle.

There are two "problems" with this equation

1. There are solutions with negative energies, which can be seen from the fact that

   $$E = \pm \sqrt{p^2 c^2 + m^2 c^4}$$

   are both valid solutions of the relativistic dispersion relation.

2. In fact the wavefunctions have negative probabilities associated with them.

Once you interpret the negative energy solutions as "antiparticles" (as Dirac later did), then the first of these problems may not seem such a big deal. But the second <u>is</u> a killer for our usual interpretation of the norm (squared) of the wavefunction as a probability density.

So what to do?

Well, we can try taking the "square root" of our operator, to isolate the positive energy solutions. This leads to something like

Note $\bar{p} \to -i\hbar\bar{\nabla}$
$E \to i\hbar \frac{\partial}{\partial t}$

$$\left(\hat{p}^0 - \sqrt{\hat{\bar{p}}^2 + m^2c^2}\right)\varphi(x^\mu) = 0$$

each of these are $\hat{p}^\mu = i\hbar \frac{\partial}{\partial x_\mu}$

Unfortunately this equation is not manifestly Lorentz covariant, because $\hat{p}^0$ and $\hat{p}^i$ are treated so asymmetrically. We want to keep things linear in $\hat{p}^0 = i\hbar \frac{\partial}{\partial t}$, to avoid negative energy solutions, and eventually this leads us to the form

$$\left(\hat{p}^0 - \alpha_1\hat{p}^1 - \alpha_2\hat{p}^2 - \alpha_3\hat{p}^3 - \beta mc\right)\varphi(x^\mu) = 0$$

The coefficients α and β must be

- dimensionless
- independent of the $\hat{p}$ operators (otherwise the operator is not linear in $\hat{p}^\mu$)
- independent of x^μ, to ensure all spacetime points are treated equally

To solve for α_i and β, we multiply by

$$\hat{p}^0 + \alpha_1 \hat{p}^1 + \alpha_2 \hat{p}^2 + \alpha_3 \hat{p}^3 + \beta mc$$

to obtain

$$\left[(\hat{p}^0)^2 - \frac{1}{2} \sum_{i,j=1}^{3} \{\alpha_i, \alpha_j\} \hat{p}^i \hat{p}^j - \sum_{i=1}^{3} \{\alpha_i, \beta\} \hat{p}^i mc - \beta^2 m^2 c^2\right] \psi(x^\mu) = 0$$

$\{\hat{a}, \hat{b}\} = \hat{a}\hat{b} + \hat{b}\hat{a}$ is the anticommutator of two operators

which we want to match to

$$\left((\hat{p}^0)^2 - (\hat{\vec{p}})^2 - m^2 c^2\right) \psi(x^\mu) = 0.$$

Thus we deduce

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}$$

$$\{\alpha_i, \beta\} = 0$$

$$\beta^2 = 1$$

It turns out there are no solutions to these equations for matrices smaller than 4×4 matrices (including numbers, that is, 1×1 matrices).

If we define

$$\beta = \gamma^0$$

$$\alpha_i = \gamma^0 \gamma^i$$

then all of these constraints can be expressed as

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$$

This relationship is called the <u>Clifford algebra</u> and the (as yet unknown) objects denoted γ^μ are the "<u>gamma matrices</u>".

One choice for the gamma matrices is

This is just one "representation" of the gamma matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

which we can represent in 2x2 block form as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

where the σ_i are the 2x2 Pauli spin matrices.

see p. 168 of Griffiths

Then we have

$$\beta = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha_i = \gamma^0 \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

Note γ^0, β, and α_i are all Hermitian, but γ^i are anti Hermitian. However, all of the gamma matrices satisfy

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$$

With these gamma matrices, our equation becomes

$$\left[\gamma^0 \hat{p}_0 + \gamma^i \hat{p}_i - mc\right] \psi(x^\mu) = 0$$

N.B. careful with indices when switching $\hat{p}^i \to \hat{p}_i$ and note the signs!

In other words

$$(\gamma^\mu \hat{p}_\mu - mc)\psi(x^\mu) = 0$$

With our operator prescription, this becomes

$$\boxed{\left(i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi(x^\mu) = 0}$$

This is the Dirac equation – isn't it beautiful ?!

Oh, and define $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$

For full beauty, use the "Feynman slash" notation $\not{a} = a_\mu \gamma^\mu$ and set $\hbar = c = 1$ to give

$$(i\not{\partial} - m)\psi(x) = 0$$

Spinors

The Dirac equation looks like a single equation, but it's really four equations in one, because the gamma matrices are 4×4 matrices. This might look inconsistent with the fact we have the term "$-mc$" in there, but you have to keep in mind that this is really $(-mc)\times \mathbb{1}_{4\times 4}$ and we usually don't bother to write the identity matrix $\mathbb{1}_{4\times 4}$.

The wavefunction $\psi(x^\mu)$ is in fact also a four-component object – a <u>spinor</u>. We can represent the spinor as a column of four components, like a vector, but it is <u>not</u> a vector. Vectors and spinors behave differently under Lorentz transformations (although the details are beyond this course).

We usually write our spinor as $\psi(x^\mu) = \begin{pmatrix} \psi^0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix}$. Spinors describe spin-$\frac{1}{2}$ particles, such as electrons and positrons.

The free particle solutions to the Dirac equation can all be written as a constant (in spacetime) spinor times a relativistic wave piece.

To see this, let's start from the equation itself

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi(x^\mu) = 0$$

and multiply on the left by $\gamma^\nu \frac{\partial}{\partial x^\nu}$ to give

$$\left(i\hbar\,\gamma^\nu\gamma^\mu \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} - mc\,\gamma^\nu\frac{\partial}{\partial x^\nu}\right)\psi(x^\mu) = 0$$

But the second term is just the first piece of the Dirac equation itself!

$$\gamma^\nu \frac{\partial}{\partial x^\nu}\psi(x^\mu) = -\frac{imc}{\hbar}\psi(x^\mu)$$

$$\Rightarrow \left(i\hbar\,\gamma^\nu\gamma^\mu\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} - mc\left(-\frac{imc}{\hbar}\right)\right)\psi(x^\mu) = 0$$

$$i\left(\gamma^\nu\gamma^\mu\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} + \frac{m^2c^2}{\hbar^2}\right)\psi(x^\mu) = 0$$

Next we use a common trick for gamma matrices

$$\gamma^\nu\gamma^\mu\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} = \gamma^\mu\gamma^\nu\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}$$

← here we switch $\mu \leftrightarrow \nu$ because they're dummy indices

$$= \gamma^\mu\gamma^\nu\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}$$

← we can reverse the order of derivatives, they commute $\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}$

So we can deduce that

$$\gamma^\nu\gamma^\mu\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} = \frac{1}{2}\left(\gamma^\nu\gamma^\mu\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} + \gamma^\mu\gamma^\nu\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\right)$$

here we've used $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$

$$= \frac{1}{2}\left(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\right)\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} = \frac{1}{2}\{\gamma^\nu,\gamma^\mu\}\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} = \frac{1}{2}\cdot 2\eta^{\mu\nu}\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}$$

But

$$\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x}\cdot\frac{\partial}{\partial x} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \overline{\nabla}^2$$

↑ four-dimensional scalar product = four dimensional analogue of the Laplacian

This operator is often denoted

$$\Box \equiv \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} \quad \text{or} \quad \partial^2 \equiv \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu}$$

Therefore our equation becomes

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\varphi(x^\mu) = 0$$

This is actually the Klein-Gordon equation again! By Fourier transforming $\varphi(x^\mu)$ as

$$\varphi(x^\mu) = \int \frac{d^4p}{(2\pi)^4} e^{ip_\mu x^\mu}\tilde{\varphi}(p^\mu)$$

we can show that the solution can be written as

$$\varphi(x^\mu) = u(\overline{p})\, e^{-i(Et - \overline{p}\cdot\overline{x})/\hbar}$$

This is a relativistic plane wave solution and the piece $u(\overline{p})$ is the spinor part of this solution. It is independent of spacetime and has four components. There is a lot of freedom in how we choose $u(\overline{p})$.

One option is to start with a particle at rest, so

$$p^\mu = (E,0,0,0) = (E,\bar{0}) \quad \Rightarrow \quad E = \pm mc^2$$

Then the Dirac equation reduces to

$$\left(i\hbar\gamma^0 \frac{\partial}{\partial t} - mc^2\right) u(\bar{0}) e^{-\frac{iEt}{\hbar}} = 0$$

$$\Rightarrow \left(i\hbar\gamma^0 (-iE/\hbar) - mc^2\right) u(\bar{0}) e^{-iEt/\hbar} = 0$$

$$\Rightarrow \gamma^0 E u(\bar{p}) = mc^2 u(\bar{0})$$

If we take $E = +mc^2$ then we end up with

$$\gamma^0 u(\bar{p}) = u(\bar{p})$$

Writing out the components of the matrix and the spinor, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\Rightarrow -u_2 = u_2 \text{ and } -u_3 = u_3 \quad \Rightarrow \quad u_2 = u_3 = 0$$

This leaves u_0 and u_1 undetermined. In fact, we can choose them to represent positive energy particles with spin up $(u_0 = 1, u_1 = 0)$ and spin down $(u_0 = 0, u_1 = 1)$.

in, say, the z direction

So $u^\uparrow(\bar{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $u^\downarrow(\bar{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

If we take $E = -mc^2$ then we end up with

$$\gamma^0 u(\bar{0}) = -u(\bar{0})$$

↖ These are the troublesome negative energy solutions that Dirac interpreted as antiparticles.

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = -\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

This tells us that $u_0 = -u_0 = 0$ and $u_1 = -u_1 = 0$, and u_2 and u_3 are undetermined. We will choose these to represent spin up and down again

So $u^{\uparrow}(\bar{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $u^{\downarrow}(\bar{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Putting this all together, we have the free particle spinor solutions (for a spin-1/2 particle at rest)

$$\psi(x^\mu) = u(\bar{0}) e^{-iEt/\hbar}$$

with

$u^{\uparrow}(\bar{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad u^{\downarrow}(\bar{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad$ for $E = +mc^2$

$u^{\uparrow}(\bar{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad u^{\downarrow}(\bar{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad$ for $E = -mc^2$

The solutions for $\bar{p} \neq 0$ are much more tricky, but if you want more information, Chapter 3 of Peskin and Schroeder's "Quantum Field Theory" is a great place to start.