Learning outcomes

You'll be able to

- Distinguish continuous and discrete symmetries
- Categorise examples as continuous or discrete
- Define the translation operator
- State Bloch's theorem and apply it to 1D systems
- Relate continuous symmetries to conservation laws and provide examples in quantum mechanics
- Define the rotation operator
- Apply commutation relations to classify operators under rotation
- Relate the parity operator and inversion symmetry
- Analyse the meaning of conserved quantities in quantum mechanical systems
- Apply your knowledge to 3D systems

Symmetries and Conservation Laws

The relationship between symmetries – transformations that leave a system invariant, or unchanged – and conservation laws – for example, conservation of energy – is one of the pillars of modern physics and one of the deepest insights we have into how the Universe works.

A.K.A. my single favourite theorem

In classical physics, Noether's theorem tells us that for every continuous symmetry, there is a conserved quantity. For example, translational invariance leads to conservation of momentum, or rotational invariance generates conservation of angular momentum.

In quantum mechanics, similar rules apply. Except now we have to be a little more careful, as we can only observe expectation values of operators, rather than, say, the wavefunction itself.

There are two types of symmetry:

discrete – transformation described by a discrete number (such as an integer)

continuous – transformation described by a continuous parameter (such as a real number)

N.B. Noether's theorem only applies to continuous symmetries.

For example, an infinite plane has continuous translational symmetry – we can move anywhere we like on that plane and it looks the same. Translations by any distance (a real number = continuous) leave the plane invariant. In contrast, an infinite chessboard (alternating black and white squares) has only discrete translational symmetry. We can only move the chessboard an integer multiple of the square size in either direction (and no intermediate distance) and still maintain invariance. This kind of discrete translational invariance is exactly the kind of symmetry exhibited by a metal, that is, a lattice of regularly arranged atoms.

Symmetries are represented mathematically by <u>operators</u>

Not so surprising now that we're working in a quantum mechanical world!

<u>Examples</u>:

operator

test function: in this case, the wavefunction

Translation operator: $\hat{T}(a)\,\varphi(x) = \varphi(x-a)$

transformed function $= \varphi'(x)$

Note the sign for translation $\varphi'(x) =$ untranslated function at $\varphi(x-a)$. The function has been moved to the right!

Parity operator: $\hat{\Pi}\,\varphi(x) = \varphi(-x)$

Rotation operator: $\hat{R}_z(\alpha)\,\varphi(r,\vartheta,\phi) = \varphi(r,\vartheta,\phi-\alpha)$

about z-axis

Translation operator

We know from classical mechanics (thanks, Emmy!) that there is an intimate relation between translation and momentum, so let's now look at the translation operator in more detail and, specifically, relate it to the momentum operator (which we know how to deal with in quantum mechanics).

First, we Taylor expand the shifted wavefunction

$$\varphi(x-a) = \sum_{n=0}^{\infty} \frac{1}{n!} (-a)^n \frac{d^n}{dx^n} \varphi(x)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-a \cdot \frac{i}{\hbar} \hat{p}\right)^n \varphi(x)$$

$$= \exp\left[-\frac{ia}{\hbar}\hat{p}\right] \varphi(x)$$

$\hat{p} = -i\hbar \frac{d}{dx}$

by definition of the exponential function

So we have

$$\hat{T}(a)\varphi(x) = \exp\left[-\frac{ia}{\hbar}\hat{p}\right]\varphi(x)$$

or

$$\boxed{\hat{T}(a) = \exp\left[-\frac{ia}{\hbar}\hat{p}\right]}$$

"momentum is the <u>generator</u> of translations"

Note that the translation operator is unitary

$$[\hat{T}(a)]^{-1} = \hat{T}(-a)$$

$$= [\hat{T}(a)]^{\dagger}$$

follows from $a \to -a$ in exponential above

So far we've just considered how wavefunctions transform (this is an example of an <u>active</u> transformation, we can think of physically "picking up the wavefunction" and moving it to the right). Operators also transform under translations (which is an example of a <u>passive</u> transformation, under which we leave the wavefunction where it is and move our coordinate system <u>in the opposite direction</u>).

We <u>define</u> the translated operator via

$$\langle \psi' | \hat{Q} | \psi' \rangle = \langle \psi | \hat{Q}' | \psi \rangle$$

untranslated operator acting on translated wavefunctions

translated operator acting on untranslated wavefunctions

The left hand side is

$$\langle \psi' | \hat{Q} | \psi' \rangle = \langle \psi | \hat{T}^{+} \hat{Q} T | \psi \rangle \qquad \text{since } \psi' = \hat{T}\psi$$

And equating this with the right hand side gives

$$\boxed{\hat{Q}' = \hat{T}^{+} \hat{Q} \hat{T}}$$

Example 6.1

Find the operator $\hat{x}'$.

To solve this, we use a "test function" $f(x)$. Using our relation $\hat{Q}' = \hat{T}^{+}\hat{Q}\hat{T}$, we have

$$\hat{x}' f(x) = (\hat{T}^{+}\hat{x}\hat{T})f(x)$$

$\hat{T}^{+}(a) = \hat{T}(-a)$

$$= [\hat{T}(-a)\,\hat{x}\,\hat{T}(a)]f(x)$$

$$= \hat{T}(-a)\,\hat{x}\,[\hat{T}(a)f(x)]$$

$\hat{T}(a)f(x) = f(x-a)$

$$= \hat{T}(-a)\,\hat{x}\,f(x-a)$$

$$= \hat{T}(-a)[x f(x-a)]$$

$\hat{T}(a)(xf(x)) = (x-a)f(x-a)$

$$= (x-(-a))\,f(x-a-(-a))$$

$$= (x+a)\,f(x)$$

Thus we deduce $\hat{x}' = \hat{x} + a$.

So we see shifting the test function to the right ($f(x) \to f(x-a)$) is the same as shifting our coordinates (or, our operator) to the left.

As a side note, $\hat{p}' = \hat{T}^{+}\hat{p}\hat{T} = \hat{p}$. The momentum operator is unchanged by translations. This is not so surprising, because the momentum operator depends only on differences in position (as a function of time $\to \frac{dx}{dt}$), not the absolute position.

Now that we have a translation operator, we can start to understand translational symmetry. There are two cases:

discrete translational symmetry $\Rightarrow$ Bloch's theorem

continuous translational symmetry $\Rightarrow$ momentum conservation

Bloch's theorem

A system is translationally invariant if the Hamiltonian is unaffected by the translation operator

$$\hat{H}' = \hat{H}$$

$$\Rightarrow \quad \hat{T}^{+}\hat{H}\hat{T} = \hat{H}$$

now left multiply by $\hat{T}$

$$\underbrace{\hat{T}\hat{T}^{+}}_{=\mathbb{1}}\hat{H}\hat{T} = \hat{T}\hat{H}$$

$$\hat{H}\hat{T} = \hat{T}\hat{H} \quad \text{or} \quad \hat{H}\hat{T} - \hat{T}\hat{H} = 0$$

$$\Rightarrow \boxed{[\hat{H},\hat{T}] = 0}$$

In practice, in one dimension, for example this means

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x) = \hat{H}' = \frac{\hat{p}'^2}{2m} + V'(x)$$

$$= \frac{\hat{p}^2}{2m} + V(x+a)$$

If a is a discrete set of numbers, then this is a discrete translation symmetry. If a is a continuous, real number, it is continu

In other words, translational symmetry means $V(x) = V(x+a)$

If two operators commute, then we can choose simultaneous eigenstates of both operators, so we can choose simultaneous eigenstates of $\hat{H}$ and $\hat{T}$

$$\hat{H}\,\psi(x) = E\,\psi(x)$$

$$\hat{T}(a)\,\psi(x) = \lambda\,\psi(x)$$

↳ eigenvalue of $\hat{T}$

$\hat{T}$ is unitary $\Rightarrow |\lambda| = 1$

$\Rightarrow$ we can choose $\lambda = e^{i\phi}$

It is convention to choose $\phi = -qa$

$$\lambda = e^{i\phi} = e^{-iqa}$$

$$\Rightarrow \hat{T}(a)\,\psi(x) = \psi(x-a) = e^{-iqa}\,\psi(x)$$

We usually express this relation $\left(\psi(x-a) = e^{-iqa}\,\psi(x)\right)$ as

$$\boxed{\psi(x) = e^{iqx}\,u(x) \quad \text{where } u(x+a) = u(x)}$$

This is Bloch's theorem - it says that the stationary states of a particle in a periodic potential (that is, $V(x+a) = V(x)$) are periodic functions ($u(x)$) times a travelling wave (e^{iqx}).

Thus, electrons in crystals have a nonzero velocity!

Momentum conservation

If $V(x+a) = V(x)$ for a any real number, then we have a continuous translational symmetry. In this case we can consider an infinitesimal translation δ and we can writ

$$\hat{T}(\delta) = e^{-i\delta\hat{p}/\hbar} \simeq 1 - \frac{i\delta}{\hbar}\hat{p}$$

$$[\hat{H}, 1 - \tfrac{i\delta}{\hbar}\hat{p}] = \overset{0}{\overset{\shortparallel}{[\hat{H}, \hat{1}]}} - \tfrac{i\delta}{\hbar}[\hat{H},\hat{p}] = \tfrac{i\delta}{\hbar}[\hat{H},\hat{p}] = 0$$

Now

$$[\hat{H}, \hat{T}(\delta)] = 0 \Rightarrow [\hat{H}, 1 - \frac{i\delta}{\hbar}\hat{p}] = 0 \Rightarrow [\hat{H}, \hat{p}] = 0$$

So: if there is a continuous translational symmetry, then the Hamiltonian commutes with the momentum operator!

But we know that when an operator commutes with the Hamiltonian, then the corresponding quantity is conserved

$$\frac{d}{dt}\langle p \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{p}] \rangle = 0$$

This is an example of a very important point

$$\boxed{\text{Continuous symmetries imply conservation laws}}$$

Note, for a single particle in a potential $V(x)$, there is only one translationally invariant potential $V(x) = V_0$ (a constant).
But this is just a free particle and so momentum conservation is kinda obvious here.
But our statement still works for more complex systems

Rotational symmetry

Recall that we wrote the translation operator as

$$\hat{T}(a) = \exp\left[-\frac{ia}{\hbar}\hat{p}\right]$$

We can use a similar logic for rotations, which act on wavefunctions as

N.B. You can ignore Section 6.7

$$\hat{R}_z(\alpha)\,\psi(r,\theta,\phi) = \psi(r,\theta,\phi-\alpha)$$

Taylor-expanding the right hand side, we have

$$\psi(r,\theta,\phi-\alpha) = \sum_{n=0}^{\infty}\frac{1}{n!}(-\alpha)^n\frac{d^n}{d\phi^n}\psi(r,\theta,\phi)$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\alpha}{\hbar}\left(-i\hbar\frac{d}{d\phi}\right)\right)^n\psi(r,\theta,\phi)$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\alpha}{\hbar}\hat{L}_z\right)^n\psi(r,\theta,\phi)$$

$$= \exp\left[-\frac{i\alpha}{\hbar}\hat{L}_z\right]\psi(r,\theta,\phi)$$

So we deduce

$$\boxed{\hat{R}_n(\alpha) = \exp\left[-\frac{i\alpha}{\hbar}\bar{n}\cdot\hat{\bar{L}}\right]}$$

Here we've generalised $\hat{R}_z(\alpha) = \exp\left[-\frac{i\alpha}{\hbar}\hat{L}_z\right]$ to three dimensions, with $\bar{n} = (x, y, z)$ and $\hat{\bar{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$

Let's look at how $\hat{r}$ and $\hat{p}$ transform under rotations?

Angular momentum is the "generator" of rotations.

We'll answer this by working with an infinitesimal rotation, δ

$$\hat{R}_z(\delta) = \exp\left[-\frac{i\delta}{\hbar}\hat{L}_z\right] \simeq 1 - \frac{i\delta}{\hbar}\hat{L}_z$$

So

$$\begin{aligned}
\hat{x}' &= \hat{R}^{\dagger}\hat{x}\hat{R} \\
&\simeq \left(1 + \frac{i\delta}{\hbar}\hat{L}_z\right)\hat{x}\left(1 - \frac{i\delta}{\hbar}\hat{L}_z\right) \\
&= \hat{x} + \frac{i\delta}{\hbar}\left[\hat{L}_z, \hat{x}\right] + \ldots \quad \text{terms proportional to } \delta^2 \\
&= \hat{x} + \frac{i\delta}{\hbar}\left(i\hbar\hat{y}\right) + \ldots \quad \text{see Equation 4.122} \\
&= \hat{x} - \delta\hat{y}
\end{aligned}$$

Similarly $\left.\begin{aligned} \hat{y}' &= \hat{y}' + \delta\hat{x} \\ \hat{z}' &= \hat{z} \end{aligned}\right\}$ we can write all three as

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} = \begin{pmatrix} 1 & -\delta & 0 \\ \delta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

and this is just the infinitesimal form of

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} = \begin{pmatrix} \cos\delta & -\sin\delta & 0 \\ \sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

for a general rotation δ (since $\cos\delta \simeq 1$ and $\sin\delta \simeq \delta$)

We can write this as $\hat{\vec{r}}\,' = R(\phi)\,\hat{\vec{r}}$

$\hat{\vec{r}}\,' = (\hat{x}', \hat{y}', \hat{z}')$

$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

This is an example of a <u>vector operator</u> (this makes sense, since we know $\vec{r}$ is a vector). In fact, this is the definition of a vector operator. Any operator $\hat{\vec{Q}}$ that behaves under rotations like

$\hat{\vec{Q}} = (\hat{Q}_1, \hat{Q}_2, \hat{Q}_3)$

$$\hat{\vec{Q}}' = R(\phi)\,\hat{\vec{Q}}$$

is, by definition, a vector operator. We say it "transforms as a vector" under rotations, and this transformation property follows from the commutator with angular momentum (because, recall, the angular momentum operator is the generator of rotations)

$$\boxed{[\hat{L}_i, \hat{Q}_j] = i\hbar\,\epsilon_{ijk}\hat{Q}_k}$$

We take this as the <u>definition</u> of a <u>vector</u> operator $\hat{Q}_j$.

<u>Examples</u>: $\hat{r}_i, \hat{p}_i, \hat{L}_i$

$[\hat{L}_i, \hat{r}_j] = i\hbar\,\epsilon_{ijk}\hat{r}_k$

$[\hat{L}_i, \hat{p}_j] = i\hbar\,\epsilon_{ijk}\hat{p}_k$

$[\hat{L}_i, \hat{L}_j] = i\hbar\,\epsilon_{ijk}\hat{L}_k$

In contrast, a scalar operator is one that is unchanged by rotations. That is, an operator that commutes with the angular momentum operator

$$\boxed{[\hat{L}_i, \hat{q}] = 0}$$

We take this as the <u>definition</u> of a <u>scalar</u> operator.

<u>Example</u>: $\hat{H}$, if it has rotational symmetry

$\Rightarrow [\hat{L}_i, \hat{H}] = 0$

Continuous rotational symmetry

For a central potential $V(\bar{r}) = V(r)$ where $r = |\bar{r}|$, then the Hamiltonian commutes with a rotation about any axis by any angle

$$[\hat{H}, \hat{R}_{\bar{n}}(\phi)] = 0$$

and if we take the angle as infinitesimal, then

$$[\hat{H}, \hat{R}_{\bar{n}}(\delta)] \simeq [\hat{H}, 1 - \frac{i\delta}{\hbar} \bar{n}.\hat{\bar{L}}]$$

$$= -\frac{i\delta}{\hbar} [\hat{H}, \bar{n}.\hat{\bar{L}}]$$

$$= 0$$

This means $[\hat{H}, \hat{L}_i] = 0$, i.e. the Hamiltonian commutes with any component of the angular momentum operator.

Again, this leads to a conservation law!

$$\frac{d}{dt} \langle \bar{L} \rangle = \langle [\hat{H}, \hat{\bar{L}}] \rangle = 0$$

$\boxed{\text{symmetries} \Rightarrow \text{conservation laws!}}$

↓

Rotational invariance leads to conservation of angular momentum!

The Hamiltonian commutes with $\hat{L}_x, \hat{L}_y, \hat{L}_z$, so it also commutes with $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \Rightarrow \hat{H}, \hat{L}_z, \hat{L}^2$ form a complete set of operators (they commute pairwise)

$$[\hat{H}, \hat{L}_z] = [\hat{H}, \hat{L}^2] = [\hat{L}_z, \hat{L}^2] = 0$$

$$\boxed{\begin{aligned} \hat{H}\varphi_{nlm} &= E_n \varphi_{nlm} \\ \hat{L}_z \varphi_{nlm} &= m\hbar \varphi_{nlm} \\ \hat{L}^2 \varphi_{nlm} &= l(l+1)\hbar^2 \varphi_{nlm} \end{aligned}}$$

We used this in the hydrogen atom, but it is true for *any* central potential.

Parity

The parity operator flips the sign of spatial coordinates. For example, in one dimension, parity acts as $x \to -x$.

So, for wavefunctions, we have

$$\hat{\Pi}\,\psi(x) = \psi(-x)$$

The parity operator is its own inverse, since

$$\hat{\Pi}\,(\hat{\Pi}\,\psi(x)) = \hat{\Pi}\,(\psi(-x)) = \psi(-(-x)) = \psi(x)$$

and, in fact, is unitary, so

$$\hat{\Pi}^{-1} = \hat{\Pi}^{+}$$

The position and momentum operators are <u>odd</u> under parity

$$\hat{x}' = \hat{\Pi}^{+}\hat{x}\,\hat{\Pi} = -\hat{x}$$

$$\hat{p}' = \hat{\Pi}^{+}\hat{p}\,\hat{\Pi} = -\hat{p}$$

and a system is <u>inversion symmetric</u> if the Hamiltonian is invariant under parity, which happens if $V(-x) = V(x)$. We can write this as

$\hat{H}' = \hat{H}$ is what we mean by inversion symmetry

$$\hat{H}' = \hat{\Pi}^{+}\hat{H}\,\hat{\Pi} = \hat{H}$$

$$\Rightarrow \hat{\Pi}\hat{H} = \underbrace{\hat{\Pi}\hat{\Pi}^{+}}_{=\mathbb{1}}\hat{H}\hat{\Pi} = \hat{H}\hat{\Pi}$$

$$\Rightarrow \hat{\Pi}\hat{H} - \hat{H}\hat{\Pi} = 0 \quad \text{or} \quad [\hat{\Pi}, \hat{H}] = 0$$

So if the Hamiltonian commutes with the parity operator, then the system is inversion symmetric.

Since the parity operator commutes with the Hamiltonian, we can choose simultaneous eigenstates of both operators

$$\hat{H}\,\varphi_n(x) = E_n\,\varphi_n(x)$$
$$\hat{\pi}\,\varphi_n(x) = \varphi_n(-x) = \pm\,\varphi_n(x)$$

The eigenvalues of $\hat{\pi}$ are restricted to ± 1, because $\hat{\pi}^2 = \mathbb{1}$.

Thus, if we have even potential, with $V(-x) = V(x)$, then the eigenstates of that system, must be either even or odd under spatial inversions. And, more specifically, since

$$\frac{d}{dt}\langle\hat{\pi}\rangle = \langle[\hat{\pi},\hat{H}]\rangle = 0,$$

parity is conserved. Thus, if our system starts in a state with even parity (so $\varphi_n(-x) = \varphi_n(x)$), then it will <u>stay</u> in an even parity state (although not necessarily the same one!).

<u>Some general comments about symmetries</u> | Sections 6.3+6.6

We've been consistently highlighting the relationship between symmetries and conservation laws. But we've been a little vague about what we mean by a "conservation law".

$\boxed{\text{Q: Why is there no conservation law for invariance under parity?}}$

There are two meanings to the existence of a "conserved quantity" (or a "conservation law").

1. The expectation value of the corresponding operator is independent of time.

2. The probability of getting any particular value (from a measurement of that quantity) is independent of time.

For the first to be true, if the operator is $\hat{Q}$ and $\frac{\partial \hat{Q}}{\partial t} = 0$ (this is a pretty reasonable requirement, since if the operator is explicitly dependent on time, then it would be unusual for its expectation value to be time-independent!), then Ehrenfest's theorem tells us that ← see Section 3.5

$$\frac{d\langle \hat{Q}\rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H},\hat{Q}]\rangle + \underbrace{\left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle}_{=0} = \frac{i}{\hbar}\langle [\hat{H},\hat{Q}]\rangle$$

So time independence of the expectation value

$$\frac{d}{dt}\langle \hat{Q}\rangle = 0$$

entails that the operator commutes with the Hamiltonian

$$\langle [\hat{H},\hat{Q}]\rangle = 0$$

It turns out that this is sufficient to guarantee the second definition! Let's prove this ...

The probability of getting a result, q_n, of a measurement of $\hat{Q}$ at time t, is given by

$$P(q_n) = |\langle \phi_n | \Psi(x,t) \rangle|^2$$

where ϕ_n is the eigenstate of $\hat{Q}$ with eigenvalue q_n

$$\hat{Q} |\phi_n\rangle = q_n |\phi_n\rangle$$

Now, we can always write the time dependence of the state as

$$\Psi(x,t) = \sum_m e^{-iE_m t/\hbar} c_m |\varphi_m(x)\rangle$$

where φ_m is the eigenstate of $\hat{H}$ with eigenvalue E_m

$$\hat{H} |\varphi_m\rangle = E_m |\varphi_m\rangle$$

Now we plug this into our expression for $P(q_n)$

$$P(q_n) = |\langle \phi_n | \sum_m e^{-iE_m t/\hbar} c_m |\varphi_m\rangle|^2$$

$$= |\sum_m e^{-iE_m t/\hbar} c_m \langle \phi_n | \varphi_m \rangle|^2$$

If $[\hat{Q}, \hat{H}] = 0$ then we can <u>always</u> find a complete set of simultaneous eigenstates of both operators. In other words, we can choose $|\phi_n\rangle = |\varphi_n\rangle$. Since the $|\varphi_n\rangle$ are eigenstates of $\hat{H}$, we can take them to be orthonormal, so $\langle \varphi_m | \varphi_n \rangle = \delta_{mn}$

Thus

$$P(q_n) = |\sum_m e^{-iE_m t/\hbar} c_m \underbrace{\langle \varphi_n | \varphi_m \rangle}_{= \delta_{nm}}|^2 = |c_n|^2$$

and this is time independent!

So we can choose whichever definition we like – they're equivalent! In both cases, if the operator satisfies $\frac{\partial \hat{Q}}{\partial t} = 0$ and $[\hat{H}, \hat{Q}]$ then the corresponding quantity is conserved.

Then final comment on symmetries is this: if $[\hat{H}, \hat{Q}] = 0$, then this usually leads to degeneracy in our quantum system. The reason is that, if $|\psi_n\rangle$ is a stationary state then $\hat{Q}|\psi_n\rangle$ will have the same energy (because it doesn't matter which order we choose, either $\hat{Q}\hat{H}$ or $\hat{H}\hat{Q}$).

More precisely:

$$\begin{aligned}
\hat{H}|\psi_n'\rangle &= \hat{H}\left(\hat{Q}|\psi_n\rangle\right) \\
&= \hat{Q}\hat{H}|\psi_n\rangle \\
&= \hat{Q}E_n|\psi_n\rangle \\
&= E_n\hat{Q}|\psi_n\rangle \\
&= E_n|\psi_n'\rangle
\end{aligned}$$

So the presence of symmetries leads (often, but not always) to degeneracies in our system. And this is the origin of our method for looking for "good states" in degenerate perturbation theory (remember those?!)

Section 7.2.2

In fact, it is not the presence of one operator that commutes with the Hamiltonian that leads to degeneracy, but multiple operators that commute with the Hamiltonian but not each other (such as $\hat{L}_x, \hat{L}_y, \hat{L}_z$ in a central potential). With sets of operators that commute with $\hat{H}$, we can form at least two sets of simultaneous eigenstates (one each for $\hat{Q}_1, \hat{Q}_2$), but we know they're distinct states because $[\hat{Q}_1, \hat{Q}_2] \neq 0$.