

Quantum Mechanics II : PHYS 314

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Overview

Prof. Chris Monahan, he/him

Email: cjmonahan@wm.edu

Class: Thursday 09:30-10:50 am Small Hall III

Office hours: Tuesday 09:30-10:50 am zoom

Webpage: cjmonahan.net

Review

Why quantum?

Even quantum mechanics
doesn't cut it at scales
smaller than an atom



The short answer is that experiments tell us that classical mechanics won't cut it at the scale of atoms and molecules.

- Black body radiation \Rightarrow electromagnetic energy is quantised
- Photoelectric effect \Rightarrow light behaves like a particle (sometimes!)
- Double-slit experiment \Rightarrow matter behaves like a wave (sometimes!)

Quantum mechanics enables

- Transistors \leftarrow otherwise no computers
- Semiconductors \leftarrow otherwise no anything these days
- MRI \leftarrow and many other medical imaging techniques
- Lasers \leftarrow otherwise no CDs! LOL, who uses those anyway?
- GPS \leftarrow thank you, atomic clocks

Classical and quantum

Classical mechanics

- describes motion of macroscopic objects at low speeds
- formulated in terms of
 - Newton's laws of motion
 - Lagrangian / Hamiltonian mechanics

(general)
but low mass! Large mass = relativity

(special)
high speeds = relativity

↑
mixing quantum mechanics
special relativity
requires quantum field theory (QFT)

Quantum mechanics

- describes motion of microscopic objects at low speeds
- formulated in terms of
 - Schrödinger equation
 - Heisenberg picture
 - Feynman path integral

↙ about the size of an atom

mixing quantum mechanics
and general relativity is
an unsolved problem

Chapter 2

Schrödinger equation describes time dependence of the wavefunction

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x,t) \Psi(x,t)$$

Interpret the wavefunction as a probability density

ie. $\int_a^b |\Psi(x,t)|^2 dx =$ probability of "finding" "particle" between a and b , at time t

that describes the state of a quantum system and "lives" in a Hilbert space.

Observables are represented by Hermitian operators in that Hilbert space and we can write our wavefunction in terms of a set of basis eigenstates of a particular operator.

$$\Psi(x,t) = \sum_n c_n(t) f_n(x)$$

Hermitian operators have real eigenvalues and orthonormal eigenstates.

The eigenvalues of an operator are the possible outcomes of a measurement, which occur with probability $|c_n|^2$.

Noncommuting operators lead to uncertainty relations. $c_n = \langle f_n | \Psi \rangle$

$$\Delta a \Delta b \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

↑ for example, $\Delta x \Delta p \geq \frac{\hbar}{2}$

For a general (time independent) quantum system, our approach is:

- write down Hamiltonian
- solve Schrödinger equation $H\Psi = E\Psi$
- use boundary conditions to determine energy eigenvalues E_n and eigenfunctions Ψ_n

↑ works very well for geometries we can solve exactly:

- infinite square well
- harmonic oscillator
- free particle
- delta function potential
- finite square well

But what happens if we can't solve the Schrödinger equation exactly (as we usually can't)?

↑ we will answer this later in the course!

Let's remind ourselves of some of the simple systems you've seen before.

Infinite square well

$$\text{Potential is } V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & \text{otherwise} \end{cases}$$

Note: my capital "Psi" is Ψ , my lowercase "psi" is ψ and my lowercase "phi" is ϕ .

Wavefunction satisfies

$$\frac{d^2 \psi}{dx^2} = -k^2 \psi \quad \text{with} \quad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

with solution

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \quad \text{if } 0 \leq x \leq a$$

and energy

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

↑ and $\psi_n(x) = 0$
otherwise

Finite square well

$$\text{Potential is } V(x) = \begin{cases} 0 & -a \leq x \leq a \\ V_0 & |x| > a \end{cases}$$

Requires matching solutions at boundaries

- wavefunction is continuous at boundary
- derivative of wavefunction is continuous at boundary, unless the potential is infinite.

↑
To find discontinuity, integrate Schrödinger equation over the boundary.

Delta function well

Potential is $V(x) = -\alpha \delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$

Allows for two types of state

1. Bound state with $E < 0$
2. Scattering state with $E > 0$

↖ Strictly speaking, a delta function is a distribution (generalised function), but it's not important for us.

One bound state

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}$$

with energy

$$E = -\frac{m\alpha^2}{2\hbar^2}$$

↙ In this system these are plane wave solutions
Scattering states allow us to define reflection and transmission coefficients, which tell us what fraction of a beam of particles are reflected or transmitted, respectively.

Free particle

Potential is $V(x) = 0$ everywhere

Plane wave solutions $\Psi(x,t) = Ae^{ik(x - \frac{\hbar k}{2m}t)} + Be^{-ik(x - \frac{\hbar k}{2m}t)}$

are not normalisable \Rightarrow they cannot be physical solutions.

Simple harmonic oscillator

Potential is $V(x) = \frac{1}{2} m \omega^2 x^2$

Remember: if in doubt,
Taylor expand until
it looks like a SHO

Wavefunction satisfies

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi_n}{dx^2} + \frac{m\omega^2 x^2}{2} \psi_n = E_n \psi_n$$

with solution

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

$$\xi^2 = \frac{m\omega}{\hbar} x^2$$

and energy

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega$$

Hermite polynomial

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2}$$

Recall two methods of solution

1. Analytic solution

2. Raising/lowering (ladder) operator method

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}} (\mp i\hat{p} + m\omega\hat{x})$$

$$\hat{a}_+ \psi_n = \sqrt{n+1} \psi_{n+1}$$

$$\hat{a}_- \psi_n = \sqrt{n} \psi_{n-1}$$

$$\hat{H}(\hat{a}_{\pm} \psi) = (E \pm \hbar\omega) \hat{a}_{\pm} \psi$$