Quantum Mechanics II: PHYS 314 (Spring 2021)
Problem Set 6–Solutions.

Overview

In this Problem Set you will revise your understanding of four vectors and look at some new solutions of the Dirac equation.

Question 1 — 20pts

An electron is travelling with a three-velocity $\mathbf{u} = (0, 0, u^z)$ in a reference frame R_1.

(a) Find the four-velocity u^μ.

(b) Suppose we transform to a reference frame, R_2, related to R_1 by a rotation of π around the x-axis. What is the four velocity in reference frame R_2?

(c) Show that the four-momentum squared, $p^2 = p^\mu p_\mu$, of the electron is invariant under the Lorentz transformation of part (b). You may express your result in terms of the mass of the electron, m_e, without substituting in the numerical value, but you should calculate the four-vector dot product explicitly.

Question 2 — 10pts

Write 250 to 300 words[1] discussing the need for, and the issues with, relativistic quantum mechanics. You should illustrate this with two examples where nonrelativistic quantum mechanics is insufficient and at least two issues with relativistic quantum mechanics (which are ultimately solved by introducing quantum field theory, but you need not mention that). Your response should be written in full sentences and addressed to a broad, but interested-in-science, audience (so think: a popular science blog post, or article in a popular science magazine, like Scientific American). You will be graded using the following rubric:

Question 3 — 20pts

(a) Show that the spinors

$$\psi_+(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot x/\hbar} u_+(p), \qquad \text{and} \qquad \psi_-(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot x/\hbar} u_-(p),$$

[1] I will count.

Aspect	Points	If you:
Motivation	4	Correctly describe two situations or systems in which relativistic quantum mechanics is required.
	2	Correctly describe one situation or system in which relativistic quantum mechanics is required
	0	Fail to motivate relativistic quantum mechanics.
Issues	4	Provide two issues with relativistic quantum mechanics.
	2	Provide one issue= with relativistic quantum mechanics.
	0	Give no issues.
Audience	2	Correctly gauge the understanding of the audience, including defining and illustrating unfamiliar terms
	1	Give a too-technical or too-simplistic explanation.

where

$$u_+(p) = \begin{pmatrix} E/c + mc \\ 0 \\ p^z \\ 0 \end{pmatrix}, \qquad \text{and} \qquad u_-(p) = \begin{pmatrix} 0 \\ E/c + mc \\ 0 \\ -p^z \end{pmatrix},$$

satisfy the Dirac equation. Suggest an interpretation for these solutions. *Hint*: Remember that $p^2 = m^2c^2$.

(b) Find the form that $u_+(p)$ and $u_-(p)$ take as the velocity of the particle approaches the speed of light. *Hint*: Do we need to worry about the mass of the particle in this limit?

(c) Show that your solutions to part (b) are eigenstates of the *helicity* operator

$$\widehat{h} = \frac{\hbar}{2}\widehat{p}^z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and find their eigenvalues.

Helicity is a well-defined property (that is, a quantum number!) of massless particles and is important in understanding and constraining high energy particle collisions at particle accelerators.

Solution 1

(a) The four-velocity u^μ satisfies $u^2 = c^2$, so the time component of the four-velocity is given by

$$c^2 = u^2 = u_0^2 - \mathbf{u}^2 = u_0^2 - (u^z)^2.$$

Solving this gives us

$$\boxed{u_0 = \pm\sqrt{c^2 + (u^z)^2}}.$$

Note: Giving only the positive root is an acceptable solution, as a negative time component of the velocity is unphysical for particles.

(b) The matrix representing the Lorentz transform that is a rotation by π around the x-axis is

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\pi & -\sin\pi \\ 0 & 0 & \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Therefore the four-velocity in the reference frame R_2 is

$$v^\mu = \Lambda^\mu_\nu u^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_0 \\ 0 \\ 0 \\ u^z \end{pmatrix} = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ -u^z \end{pmatrix}.$$

In other words

$$\boxed{v^\mu = (\sqrt{c^2 + (u^z)^2}, 0, 0, -u^z)}.$$

(c) In reference frame R_1 the momentum squared is

$$p^2 = (m_e u^\mu)(m_e u_\mu) = m_e^2 \left(\sqrt{c^2 + (u^z)^2} \cdot \sqrt{c^2 + (u^z)^2} - u^z \cdot u^z \right) = m_e^2 c^2.$$

In frame R_1, the momentum squared is

$$q^2 = (m_e v^\mu)(m_e v_\mu) = m_e^2 \left(\sqrt{c^2 + (-u^z)^2} \cdot \sqrt{c^2 + (-u^z)^2} - (-u^z) \cdot (-u^z) \right) = m_e^2 c^2.$$

This is clearly invariant!

Note: for full marks, it is not sufficient to just use $u^2 = v^2 = c^2$ and skip the explicit calculation of the four-vector dot product.

Solution 2

See the rubric in the question.

Solution 3

(a) The Dirac equation is

$$\left[i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right]\psi(x) = 0.$$

Plugging in our solutions, we have

$$\left[i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right]\int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot x/\hbar} u_\pm(p) = 0$$

Carrying out the derivatives gives us

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\left[i\hbar\gamma^\mu\left(-\frac{ip_\mu}{\hbar}\right) - mc\right] e^{-ip\cdot x/\hbar} u_\pm(p) = 0,$$

which must hold for any x, so

$$\left[\gamma^\mu p_\mu - mc\right] u_\pm(p) = 0.$$

Writing out the matrices explicitly, we have

$$\left[\gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3 - mc\,\mathbb{I}\right] u_\pm(p) = 0.$$

Based on the form of $u_\pm(p)$, we can see that $p^\mu = (E/c, 0, 0, p^z)$, so our equation reduces to

$$\left[\gamma^0 E - \gamma^3 p^z - mc\,\mathbb{I}\right] u_\pm(p) = 0.$$

Now let's plug in an explicit representation of the gamma matrices (we'll use the one from the lecture notes), to give

$$\begin{pmatrix} E/c - mc & 0 & -p^z & 0 \\ 0 & E/c - mc & 0 & +p^z \\ p^z & 0 & -E/c - mc & 0 \\ 0 & -p^z & 0 & -E/c - mc \end{pmatrix} u_\pm(p) = 0.$$

Taking each of our solutions in turn, we have

$$\begin{pmatrix} E/c - mc & 0 & -p^z & 0 \\ 0 & E/c - mc & 0 & +p^z \\ p^z & 0 & -E/c - mc & 0 \\ 0 & -p^z & 0 & -E/c - mc \end{pmatrix} \begin{pmatrix} E/c + mc \\ 0 \\ p^z \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} (E/c - mc)(E/c + mc) - (p^z)^2 \\ 0 \\ p^z(E/c + mc) - (E/c + mc)p^z \\ 0 \end{pmatrix}.$$

It is obvious that $u_+^3 = 0$, so now we just need to do some algebra for u_+^0. We have

$$\begin{aligned}(E/c - mc)(E/c + mc) - (p^z)^2 &= \frac{E^2}{c^2} - m^2c^2 - (p^z)^2 \\ &= \frac{m^2c^4 + (p^z)^2c^2}{c^2} - m^2c^2 - (p^z)^2 \\ &= 0.\end{aligned}$$

Therefore $u_+(p)$ satisfies the Dirac equation.

Similarly

$$\begin{pmatrix} E/c - mc & 0 & -p^z & 0 \\ 0 & E/c - mc & 0 & +p^z \\ p^z & 0 & -E/c - mc & 0 \\ 0 & -p^z & 0 & -E/c - mc \end{pmatrix} \begin{pmatrix} 0 \\ E/c + mc \\ 0 \\ -p^z \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ (E/c - mc)(E/c + mc) - (p^z)^2 \\ 0 \\ -p^z(E/c + mc) - (E/c + mc)p^z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So $u_-(p)$ also satisfies the Dirac equation!

(b) In this limit we can neglect the mass and the energy becomes $E/c \approx p^z$. So our solutions become

$$u_+(p) \approx \begin{pmatrix} p^z \\ 0 \\ p^z \\ 0 \end{pmatrix} \qquad \text{and} \qquad u_-(p) = \begin{pmatrix} 0 \\ p^z \\ 0 \\ -p^z \end{pmatrix}.$$

(c) The helicity operator acts on these large momentum solutions as

$$\widehat{h}u_+(p) = \frac{\hbar\widehat{p}^z}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} p^z \\ 0 \\ p^z \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} p^z \\ 0 \\ p^z \\ 0 \end{pmatrix},$$

and

$$\widehat{h}u_-(p) = \frac{\hbar\widehat{p}^z}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ p^z \\ 0 \\ -p^z \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ -p^z \\ 0 \\ p^z \end{pmatrix}.$$

The eigenvalues are

$$\boxed{\lambda_\pm = \pm\frac{\hbar}{2}},$$

for $u_\pm(p)$ in the high energy limit.