Quantum Mechanics II: PHYS 314 (Spring 2021)
Problem Set 4–Solutions.

Overview

In this Problem Set you will apply your understanding of the relationship between spin and symmetry to construct wavefunctions for simple multiparticle systems and then apply the free electron gas model to white dwarf stars.

Question 1 [Griffiths 5.8] 20pts

Suppose you had *three* particles, one in state $\psi_a(x)$, one in state $\psi_b(x)$, and one in state $\psi_c(x)$. Assuming ψ_a, ψ_b, and ψ_c are orthonormal, construct the three-particle states analogous to Equations 5.19, 5.20, and 5.21, which are

$$\begin{aligned}
\psi(x_1, x_2) &= \psi_a(x_1)\psi_b(x_2), \\
\psi(x_1, x_2) &= \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) + \psi_a(x_2)\psi_b(x_1)\right], \\
\psi(x_1, x_2) &= \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) - \psi_a(x_2)\psi_b(x_1)\right],
\end{aligned}$$

representing

(a) distinguishable particles,

(b) identical bosons, and

(c) identical fermions.

Keep in mind that (b) must be completely symmetric, under interchange of *any* pair of particles, and (c) must be completely *anti*-symmetric, in the same sense. *Comment*: There's a cute trick for constructing completely antisymmetric wavefunctions: Form the **Slater determinant**, whose first row is $\psi_a(x_1)$, $\psi_b(x_1)$, $\psi_c(x_1)$, etc., whose second row is $\psi_a(x_2)$, $\psi_b(x_2)$, $\psi_c(x_2)$, etc., and so on (this devices works for any number of particles).

Solution 1

The wavefunctions are

(a)

$$\psi(x_1, x_2, x_3) = \psi_a(x_1)\psi_b(x_2)\psi_c(x_3)$$

(b)

$$\psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}\Big[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) + \psi_b(x_1)\psi_a(x_2)\psi_c(x_3) + \psi_c(x_1)\psi_b(x_2)\psi_a(x_3)$$
$$+ \psi_a(x_1)\psi_c(x_2)\psi_b(x_3) + \psi_b(x_1)\psi_c(x_2)\psi_a(x_3) + \psi_c(x_1)\psi_a(x_2)\psi_b(x_3)\Big]$$

(c)

$$\psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}\begin{vmatrix} \psi_a(x_1), & \psi_b(x_1), & \psi_c(x_1) \\ \psi_a(x_2), & \psi_b(x_2), & \psi_c(x_2) \\ \psi_a(x_3), & \psi_b(x_3), & \psi_c(x_3) \end{vmatrix}$$
$$= \frac{1}{\sqrt{6}}\Big[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) - \psi_b(x_1)\psi_a(x_2)\psi_c(x_3) - \psi_c(x_1)\psi_b(x_2)\psi_a(x_3)$$
$$- \psi_a(x_1)\psi_c(x_2)\psi_b(x_3) + \psi_b(x_1)\psi_c(x_2)\psi_a(x_3) + \psi_c(x_1)\psi_a(x_2)\psi_b(x_3)\Big]$$

Note: Both forms are acceptable for full marks.

Question 2 [Griffiths 5.35] 30pts

Certain cold stars (called **white dwarfs**) are stabilized against gravitational collapse by the degeneracy pressure of their electrons (Equation 5.57)

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3}.$$

Assuming constant density, the radius R of such an object can be calculated as follows:

(a) Write the total electron energy (Equation 5.56)

$$E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nd)^{5/3}}{10\pi^2 m}V^{-2/3},$$

in terms of the radius, the number of nucleons (protons and neutrons) N, the number of electrons per nucleon d, and the mass of the electron m. *Beware*: In this problem we are recycling the letters N and d for a slightly different purpose than in the text.

(b) Look up, or calculate, the gravitational energy of a uniformly dense sphere. Express your answer in terms of G (the constant of universal gravitation), R, N, and M (the mass of a nucleon). Note that the gravitational energy is *negative*.

(c) Find the radius for which the total energy, (a) plus (b), is a minimum. *Answer*:

$$R = \left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2 d^{5/3}}{GmM^2N^{1/3}}.$$

(Note that the radius *decreases* as the total mass *increases*!) Put in the actual numbers, for everything except N, using $d = 1/2$ (actually, d decreases a bit as the atomic number increases, but this is close enough for our purposes). *Answer*: $R = 7.6 \times 10^{25}\ \mathrm{N}^{-1/3}\mathrm{m}$.

(d) Determine the radius, in kilometres, of a white dwarf with the mass of the sun.

(e) Determine the Fermi energy, in electron volts, for the white dwarf in (d), and compare it with the rest energy of an electron. Note that this system is getting dangerously relativistic (see Problem 5.36).

Solution 2

(a) The volume is

$$V = \frac{4}{3}\pi R^3,$$

so

$$E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nd)^{5/3}}{10\pi^2 m}\left(\frac{4}{3}\pi R^3\right)^{-2/3} = \boxed{\frac{2\hbar^2}{15\pi m R^2}\left(\frac{9\pi Nq}{4}\right)^{5/3}}.$$

(b) To solve this, we imagine building up a sphere by adding layers of thickness $\mathrm{d}r$. When the sphere has a mass m, and radius r, the work necessary to bring in the next layer (of mass $\mathrm{d}m$) is

$$\mathrm{d}W = -\frac{Gm}{r}\mathrm{d}m.$$

Now we want to express this in terms of the density, which is related to the mass by $m = 4\pi r^3/3 \cdot \rho$, from which we deduce

$$\mathrm{d}m = 4\pi r^2 \rho \mathrm{d}r.$$

Then the work we have to do to increase the mass is

$$\mathrm{d}W = -\frac{Gm}{r}\cdot\left(4\pi r^2\rho\mathrm{d}r\right) = -\frac{G}{r}\cdot\left(4\pi r^3/3\cdot\rho\right)\cdot\left(4\pi r^2\rho\mathrm{d}r\right) = -\frac{16\pi^3}{3}\rho^2 G r^4 \mathrm{d}r.$$

Thus the total energy of the sphere is

$$E_{\text{grav}} = -\frac{16\pi^3}{3}\rho^2 G\int_0^R r^4\mathrm{d}r = -\frac{16\pi^3}{3}\rho^2 G\frac{R^5}{5}.$$

But we can express the density as the total mass of N nucleons, divided by their volume:

$$\rho = \frac{NM}{4\pi R^3/3}.$$

Plugging this into our expression for the energy gives

$$E_{\text{grav}} = -\frac{16\pi^2 R^5 G}{15} \cdot \left(\frac{3NM}{4\pi R^3}\right)^2 = \boxed{-\frac{3GN^2M^2}{5R}}.$$

(c) We need the total energy, which is the sum of our two contributions (where A and B are defined to match the solutions of parts (a) and (b))

$$E_{\text{tot}} = \frac{A}{R^2} - \frac{B}{R}.$$

This has a minimum at

$$\frac{\mathrm{d}}{\mathrm{d}R} E_{\text{tot}} = -\frac{A}{2R^3} + \frac{B}{R^2} = 0,$$

which has solution $R = 2A/B$. If we plug in our values for A and B, we have

$$R = \frac{2A}{B} = 2\left(\frac{2\hbar^2}{15\pi m R^2}\left(\frac{9\pi N q}{4}\right)^{5/3}\right)\left(\frac{3GN^2M^2}{5R}\right)^{-1} = \boxed{\left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2 q^{5/3}}{GmM^2N^{1/3}}}.$$

Plugging in some numbers, we have

$$\begin{aligned} R &= \left(\frac{9\pi}{4}\right)^{2/3} \frac{(1.033 \times 10^{-34}\ \mathrm{J \cdot s})^2 (1/2)^{5/3}}{(6.673 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(9.109 \times 10^{-31}\ \mathrm{kg})(1.674 \times 10^{-27}\ \mathrm{kg})^2 N^{1/3}} \\ &= \boxed{\left(7.58 \times 10^{25}\ \mathrm{m}\right) N^{-1/3}}. \end{aligned}$$

(d) The mass of the sun is $M_\odot = 1.989 \times 10^{30}$ kg. Thus the total number of nucleons is

$$N = \frac{M_\odot}{M} = \frac{1.989 \times 10^{30}\ \mathrm{kg}}{1.674 \times 10^{-27}\ \mathrm{kg}} = 1.188 \times 10^{57},$$

which is a lot of nucleons. Thus the radius would be

$$R = \left(7.58 \times 10^{25}\ \mathrm{m}\right)(1.188 \times 10^{57})^{-1/3} = \boxed{7.16 \times 10^6\ \mathrm{m}}.$$

This is slightly larger than the Earth.

(e) The Fermi energy is given by [Equation 5.54]

$$\begin{aligned}
E_F &= \frac{\hbar^2}{2mR^2}\left(\frac{9\pi Nq}{4}\right)^{2/3} \\
&= \frac{(1.055\times 10^{-34}\,\mathrm{J\cdot s})^2}{2(9.109\times 10^{-31}\,\mathrm{kg})(7.16\times 10^{6}\,\mathrm{m})}\left(\frac{9\pi(1.188\times 10^{57})(1/2)}{4}\right)^{2/3} \\
&= 3.102\times 10^{-14}\,\mathrm{J} \\
&= \frac{3.102\times 10^{-14}\,\mathrm{J}}{1.602\times 10^{-19}\,\mathrm{J/eV}} = \boxed{1.94\times 10^{5}\,\mathrm{eV}}.
\end{aligned}$$

Let's compare this to the rest energy (that is, the mass of an electron), which is given by

$$E_{\text{rest}} = mc^2 = 5.11\times 10^{5}\,\mathrm{eV}.$$

Thus

$$\frac{E_F}{E_{\text{rest}}} = \frac{1.94\times 10^{5}\,\mathrm{eV}}{5.11\times 10^{5}\,\mathrm{eV}} = 0.38.$$

This is a significant fraction of the rest energy, and the system is therefore pretty relativistic.