Quantum Mechanics II: PHYS 314 (Spring 2021)
Problem Set 1–Solutions.

Overview

In this Problem Set you will revise material you covered in Quantum Mechanics I, including reminding yourself of the properties of the infinite square well, how to calculate expectation values, and reviewing properties of eigenfunctions.

Question 1 [Griffiths 2.4] 20pts

Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, σ_x and σ_p for the n^{th} stationary state of the infinite square well. Check that the uncertainty principle is satisfied. Which state comes closest to the uncertainty limit?

Solution 1

We need the wavefunctions for the infinite square well, which are given by Equation 2.31,

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}\right).$$

Then we can calculate the expectation values through the appropriate integrals. We start with $\langle x\rangle$,

$$\langle x\rangle = \int_0^a \mathrm{d}x\, x\,|\psi_n(x)|^2 = \int_0^a \mathrm{d}x\, x\, \sin^2\left(\frac{n\pi}{a}x\right) = \boxed{\frac{a}{2}},$$

which is independent of n. In other words, particles in all states are likely to be found in the middle of the box.

The next is

$$\langle x^2\rangle = \int_0^a \mathrm{d}x\, x^2\,|\psi_n(x)|^2 = \int_0^a \mathrm{d}x\, x^2\, \sin^2\left(\frac{n\pi}{a}x\right) = \boxed{a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right]}.$$

Then we have

$$\langle p\rangle = m\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle = \boxed{0}, \tag{1}$$

and

$$\begin{aligned}
\langle p^2\rangle &= \int_0^a \mathrm{d}x\, \psi_n^*(x) \left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 \psi_n(x) \\
&= -\hbar^2 \int_0^a \mathrm{d}x\, \psi_n^*(x) \left(-\frac{2mE_n}{\hbar^2}\psi_n(x)\right) \\
&= 2mE_n \int_0^a \mathrm{d}x\, |\psi_n^*(x)|^2 \\
&= 2mE_n,
\end{aligned}$$

where we used the time-independent Schrödinger equation instead of directly calculating the second derivative of the wavefunction. Thus

$$\boxed{\langle p^2\rangle = \left(\frac{n\pi\hbar}{a}\right)^2}.$$

Now we're in a position to calculate the uncertainties. We have

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2} - \frac{1}{4}\right]} = \boxed{\frac{a}{2}\sqrt{\frac{1}{3} - \frac{2}{(n\pi)^2}}},$$

and

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\left(\frac{n\pi\hbar}{a}\right)^2} = \boxed{\frac{n\pi\hbar}{a}}.$$

The product of these gives

$$\sigma_x\sigma_p = \frac{a}{2}\sqrt{\frac{1}{3} - \frac{1}{2(n\pi)^2}}\frac{n\pi\hbar}{a} = \boxed{\frac{\hbar}{2}\sqrt{\frac{(n\pi)^2}{3} - 2}}.$$

To check that this is always greater than $\hbar/2$, we can see that the minimum value of $\sigma_x\sigma_p$ occurs for $n = 1$, in which case we have

$$\sigma_x\sigma_p\Big|_{n=1} = \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3} - 2} \simeq 1.136\frac{\hbar}{2}.$$

This is clearly larger than $\hbar/2$, so the uncertainty principle is *satisfied* and the state that comes closes to the bound is the *ground state.*

Question 2 [Griffiths 3.7] **15pts**

(a) Suppose that $f(x)$ and $g(x)$ are two eigenfunctions of an operator $\widehat{Q}$, with the same eigenvalue q. Show that any linear combination of f and g is itself an eigenfunction of $\widehat{Q}$, with eigenvalue q.

(b) Check that $f(x) = \exp(x)$ and $g(x) = \exp(-x)$ are eigenfunctions of the operator $\mathrm{d}^2/\mathrm{d}x^2$, with the same eigenvalue. Construct two linear combintations of f and g that are *orthogonal* eigenfunctions on the interval $(-1, 1)$.

Solution 2

(a) We know that

$$\widehat{Q}f(x) = qf(x), \qquad \text{and} \qquad \widehat{Q}g(x) = qg(x),$$

so let's define an arbitrary linear combination of these functions, $h(x) = af(x)+bg(x)$, and apply the operator to that combination. We have

$$\begin{aligned}\widehat{Q}h(x) &= \widehat{Q}[af(x) + bg(x)] \\ &= a\widehat{Q}f(x) + b\widehat{Q}g(x) \\ &= aqf(x) + bqg(x) \\ &= q[af(x) + bg(x)].\end{aligned}$$

Thus

$$\boxed{\widehat{Q}h(x) = qh(x)},$$

as required for $h(x)$ to be an eigenfunction of $\widehat{Q}$.

(b) In this case the operator is $\widehat{Q} = \mathrm{d}^2/\mathrm{d}x^2$ and our functions are $f(x) = \exp(x)$ and $g(x) = \exp(-x)$. Let's apply the operator to these and see what happens. We have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\exp(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}\exp(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = \exp(x),$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\exp(-x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}\exp(-x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}[-\exp(-x)] = \exp(-x),$$

so clearly both these functions are eigenfunctions of $\widehat{Q} = \mathrm{d}^2/\mathrm{d}x^2$, with eigenvalue one.

Now we need to construct orthogonal combinations of these functions. The simplest options are $N(f(x) \pm g(x))$, with N some normalisation. In this case we have

$$f(x) \pm g(x) = e^x \pm e^{-x}.$$

We are not asked for orthonormal solutions, so we don't need to worry about the normalisation, in which case we might as well choose

$$h_1(x) = \frac{e^x + e^{-x}}{2} = \sinh(x), \tag{2}$$

and

$$h_2(x) = \frac{e^x - e^{-x}}{2} = \cosh(x). \tag{3}$$

These solutions are clearly orthogonal, because $\sinh(x)$ is odd, while $\cosh(x)$ is even. **Note that other solutions are acceptable, provided they are demonstrably orthogonal combinations of e^x and e^{-x}.**

Question 3 [Griffiths 3.12] 15pts

Find $\Phi(p,t)$, defined through the inverse Fourier transform of the wavefunction [Equation 3.54],

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ipx/\hbar} \Psi(x,t),$$

for the *free particle*, in terms of the function $\phi(k)$ introduced in Equation 2.101, which is

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k\, \phi(k) e^{i(kx - \hbar^2 k^2 t/(2m))}.$$

Show that for the free particle $|\Phi(p,t)|^2$ is independent of time.

Solution 3

There are two ways to do this. The first is to plug Equation 2.101 into Equation 3.54 and reverse the order of integration and carry out the integrals. The second is to rewrite $k = p/\hbar$ in Equation 2.101, which becomes

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\hbar} \phi\left(\frac{p}{\hbar}\right) e^{i(px/\hbar - p^2 t/(2m))} \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}p\, e^{ipx/\hbar} \left[\frac{1}{\sqrt{\hbar}} \phi\left(\frac{p}{\hbar}\right) e^{-ip^2 t/(2m)}\right].\end{aligned}$$

By comparing this to Equation 3.55, which is

$$\Psi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{ipx/\hbar} \Phi(p,t),$$

we see that

$$\boxed{\Phi(p,t) = \frac{1}{\sqrt{\hbar}} \phi\left(\frac{p}{\hbar}\right) e^{-ip^2 t/(2m)}}.$$

We can now calculate $|\Phi(p,t)|^2$, which is

$$\boxed{|\Phi(p,t)|^2 = \frac{1}{\hbar}\left|\phi\left(\frac{p}{\hbar}\right)\right|^2},$$

and clearly time independent, as required.