Quantum Mechanics II: PHYS 314 (Spring 2021)
Problem Set 9–Due Thursday April 29.

Overview

In this Problem Set–the last one!–you will apply your knowledge of first-order time-dependent perturbation theory to a two-level systems, and to a time-dependent variant of the infinite square well, and study the so-called "quantum Zeno effect".

Question 1 [Griffiths 11.8] **15pts**

Consider a perturbation to a two-level system with matrix elements

$$H'_{ab} = H'_{ba} = \frac{\alpha}{\sqrt{\pi}\tau} e^{-(t/\tau)^2}, \qquad H'_{aa} = H'_{bb} = 0,$$

where τ and α are positive constants with the appropriate units.

(a) According to first-order perturbation theory, if the system starts off in the state $c_a = 1$, $c_b = 0$ at $t = -\infty$, what is the probability that it will be found in the state b at $t = \infty$?

(b) In the limit that $\tau \to 0$, $H'_{ab} = \alpha\delta(t)$. Compute the $\tau \to 0$ limit of your expression from part (a) and compare to the exact result [which is calculated in Problem 11.4]

$$P_{a\to b} = \sin^2\left(\frac{\alpha}{\hbar}\right).$$

(c) Now consider the opposite extreme: $\omega_0\tau \gg 1$. What is the limit of your expression from part (a)? *Comment*: This is an example of the adiabatic theorem (Section 11.5.2).

Question 2 [Griffiths 11.27] **10pts**

A particle of mass m is initially in the ground state of the (one-dimensional) infinite square well. At time $t = 0$ a "brick" is dropped into the well, so that the potential becomes

$$V(x) = \begin{cases} V_0 & 0 \le x \le a/2, \\ 0 & a/2 < x \le a, \\ \infty & \text{otherwise}, \end{cases}$$

where $V_0 \ll E_1$. After a time T the brick is removed, and the energy of the particle is measured. Find the probability (in first-order perturbation theory) that the result is now E_2.

Question 3 — 25pts

A quantum system has just two energy eigenstates, ψ_1 and ψ_2, with corresponding eigenvalues E_1 and E_2. Assume that $E_2 > E_1$. The states are also characterised by parity, which is represented by an operator $\widehat{\mathcal{P}}$ that acts on the energy eigenstates as

$$\widehat{\mathcal{P}}|\psi_1\rangle = |\psi_2\rangle, \qquad \text{and} \qquad \widehat{\mathcal{P}}|\psi_2\rangle = |\psi_1\rangle.$$

(a) Find the eigenstates of the parity operator in terms of $|\psi_1\rangle$ and $|\psi_2\rangle$.

(b) Assuming that the system is initially in a positive-parity eigenstate, find the state of the system at any later time, $t > 0$.

(c) At a particular time T a parity measurement is made on the system. What is the probability of finding the system with positive parity?

(d) Imagine that instead of a single measurement at time T you make a series of N parity measurements at the times Δt, $2\Delta t$, and so on, up to $N\Delta t = T$. Assuming that N is very large and $\Delta t \ll (E_2 - E_1)/\hbar$, what is the probability of finding the system with positive parity at time T? Compare this probability with the probability of finding the system in the positive parity state with a single measurement at $t = T$ (that is, your answer to part (c)). This "freezing" of the system in the initial state for a repeated series of measurements has been called the "quantum Zeno effect". *Hint*: You may find the series expansion

$$\left(1 - \frac{x^2}{n^2}\right)^n \simeq e^{-x^2/n}$$

useful. [*Note*: see this Wikipedia page for a quick primer on Zeno's paradox(es), if you are not familiar with the motivation for this name. You can find more information on the quantum Zeno effect on this Wikipedia page.]