Quantum Mechanics II: PHYS 314 (Spring 2021)
Problem Set 6–Due Thursday March 25.

In this Problem Set you will revise your understanding of four vectors and look at some new solutions of the Dirac equation.

Question 1 **20pts**

An electron is travelling with a three-velocity $\mathbf{u} = (0, 0, u^z)$ in a reference frame R_1.

(a) Find the four-velocity u^μ.

(b) Suppose we transform to a reference frame, R_2, related to R_1 by a rotation of π around the x-axis. What is the four velocity in reference frame R_2?

(c) Show that the four-momentum squared, $p^2 = p^\mu p_\mu$, of the electron is invariant under the Lorentz transformation of part (b). You may express your result in terms of the mass of the electron, m_e, without substituting in the numerical value, but you should calculate the four-vector dot product explicitly.

Question 2 **10pts**

Write 250 to 300 words[1] discussing the need for, and the issues with, relativistic quantum mechanics. You should illustrate this with two examples where nonrelativistic quantum mechanics is insufficient and at least two issues with relativistic quantum mechanics (which are ultimately solved by introducing quantum field theory, but you need not mention that). Your response should be written in full sentences and addressed to a broad, but interested-in-science, audience (so think: a popular science blog post, or article in a popular science magazine, like Scientific American). You will be graded using the following rubric:

[1] I will count.

Aspect	Points	If you:
Motivation	4	Correctly describe two situations or systems in which relativistic quantum mechanics is required.
	2	Correctly describe one situation or system in which relativistic quantum mechanics is required
	0	Fail to motivate relativistic quantum mechanics.
Issues	4	Provide two issues with relativistic quantum mechanics.
	2	Provide one issue= with relativistic quantum mechanics.
	0	Give no issues.
Audience	2	Correctly gauge the understanding of the audience, including defining and illustrating unfamiliar terms
	1	Give a too-technical or too-simplistic explanation.

Question 3 — 20pts

(a) Show that the spinors

$$\psi_+(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot x/\hbar} u_+(p), \qquad \text{and} \qquad \psi_-(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot x/\hbar} u_-(p),$$

where

$$u_+(p) = \begin{pmatrix} E/c + mc \\ 0 \\ p^z \\ 0 \end{pmatrix}, \qquad \text{and} \qquad u_-(p) = \begin{pmatrix} 0 \\ E/c + mc \\ 0 \\ -p^z \end{pmatrix},$$

satisfy the Dirac equation. Suggest an interpretation for these solutions. *Hint*: Remember that $p^2 = m^2c^2$.

(b) Find the form that $u_+(p)$ and $u_-(p)$ take as the velocity of the particle approaches the speed of light. *Hint*: Do we need to worry about the mass of the particle in this limit?

(c) Show that your solutions to part (b) are eigenstates of the *helicity* operator

$$\widehat{h} = \frac{\hbar}{2}\widehat{p}^z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and find their eigenvalues.

Helicity is a well-defined property (that is, a quantum number!) of massless particles and is important in understanding and constraining high energy particle collisions at particle accelerators.