Quantum Mechanics II: PHYS 314 (Spring 2021)
Problem Set 3–Due Thursday, February 25.

Overview

In this Problem Set you will practice manipulating operators and operator exponentials by studying the parity operator in three dimensions, and by investigating Galilean transformations in quantum mechanical systems. Part (d) of Problem 2 is for **extra credit**. You do **not** need to complete this part of question to receive full marks for the Problem Set.

Question 1 [Griffiths 6.1] 20pts

Consider the parity operator in three dimensions.

(a) Show that $\psi'(\mathbf{r}) = \widehat{\Pi}\,\psi(\mathbf{r}) = \psi(-\mathbf{r})$ is equivalent to a mirror reflection followed by a rotation.

(b) Show that, for ψ expressed in polar coordinates, the action of the parity operator is

$$\widehat{\Pi}\,\psi(r,\theta,\phi) = \psi(r,\pi-\theta,\phi+\pi).$$

(c) Show that for the hydrogenic orbitals,

$$\widehat{\Pi}\,\psi_{n\ell m}(r,\theta,\phi) = (-1)^{\ell}\psi(r,\theta,\phi).$$

That is, $\psi_{n\ell m}$ is an eigenstate of the parity operator, with eigenvalue $(-1)^{\ell}$.

Question 2 [Griffiths 6.35] 30pts

A **Galilean transformation** performs a boost from a reference frame $\mathcal{S}$ to a reference frame $\mathcal{S}'$ moving with velocity $-v$ with respect to $\mathcal{S}$ (the origins of the two frames coincide at $t=0$). The unitary operator that carries out a Galilean transformation at time t is

$$\widehat{\Gamma}(v,t) = \exp\left[-\frac{i}{\hbar}v\,(t\widehat{p} - m\widehat{x})\right].$$

(a) Find $\widehat{x}' = \widehat{\Gamma}^{\dagger}\widehat{x}\widehat{\Gamma}$ and $\widehat{p}' = \widehat{\Gamma}^{\dagger}\widehat{p}\widehat{\Gamma}$ for an infinitesimal transformation with velocity δ. What is the physical meaning of your result?

(b) Show that

$$\widehat{\Gamma}(v,t) = \exp\left[\frac{i}{\hbar}\left(mv\widehat{x} - \frac{mv^2}{2}t\right)\right]\widehat{T}(vt)$$
$$= \widehat{T}(vt)\exp\left[\frac{i}{\hbar}\left(mv\widehat{x} + \frac{mv^2}{2}t\right)\right],$$

where $\widehat{T}$ is the spatial translation operator (Equation 6.3),

$$\widehat{T}(a) = \exp\left[-\frac{ia}{\hbar}\widehat{p}\right].$$

You will need to use the Baker-Campbell-Hausdorff formula (Problem 3.29)

$$\widehat{\Gamma} = e^{\widehat{A}+\widehat{B}} = e^{\widehat{A}}e^{\widehat{B}}e^{-\widehat{C}/2}.$$

(c) Show that if Ψ is a solution to the time-dependent Schrödinger equation with Hamiltonian

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + V(x)$$

then the boosted wavefunction $\Psi' = \widehat{\Gamma}(v,t)\Psi$ is a solution to the time-dependent Schrödinger equation with the potential $V(x)$ in motion:

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + V(x - vt).$$

Note:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{\widehat{A}} = e^{\widehat{A}}\frac{\mathrm{d}\widehat{A}}{\mathrm{d}t}$$

only if

$$\left[\widehat{A}, \frac{\mathrm{d}\widehat{A}}{\mathrm{d}t}\right] = 0.$$

(d) **[EXTRA CREDIT: 5 PTS]**

Show that the result of Problem 2.50a,

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x-vt|/\hbar^2}\exp\left\{-\frac{i}{\hbar}\left[\left(E + \frac{mv^2}{2}\right)t - mvx\right]\right\},$$

for a moving delta-function well $V(x,t) = -\alpha\delta(x - vt)$, is an example of this result [i.e. the result of part (c)].