Quantum Mechanics II: PHYS 314 (Spring 2021)
Midterm 2–Prep questions.

Question 1 [Griffiths 10.4]

Consider the case of low-energy scattering from a spherical delta-function shell:

$$V(r) = \alpha\delta(r-a),$$

where α and a are positive constants. Calculate the scattering amplitude, $f(\theta)$, the differentical cross-section, $D(\theta)$, and the total cross-section, σ. Assume $ka \ll 1$, so that only the $\ell = 0$ term contributes significantly. (To simplify matters, throw out all $\ell \neq 0$ terms right from the start.) The main problem, of course, is to determine C_0. Express your answer in terms of the dimensionless quantity $\beta \equiv 2ma\alpha/\hbar^2$. *Answer*: $\sigma = 4\pi a^2\beta^2/(1+\beta)^2$.

Solution 1

Keeping only the $\ell = 0$ terms, we use Equation 10.29, which is

$$\psi(r,\theta) = A\sum_{\ell=0}^{\infty} i^\ell(2\ell+1)\left[j_\ell(kr) + ika_\ell h_\ell^{(1)}(kr)\right]P_\ell(\cos\theta).$$

In the exterior region, this is

$$\begin{aligned}\psi(r) &\simeq A\left[j_0(kr) + ika_0 h_0^{(1)}(kr)\right]P_0(\cos\theta)\\ &= A\left[\frac{\sin(kr)}{kr} + ika_0\left(-i\frac{e^{ikr}}{kr}\right)\right]\\ &= A\left[\frac{\sin(kr)}{kr} + \frac{a_0e^{ikr}}{r}\right],\end{aligned}$$

for $r > a$.

In the interior region, we need to be careful with the n_ℓ piece, because it blows up at the origin. Using Equation 10.18, which is

$$u(r) = Arj_\ell(kr) + Brn_\ell(kr),$$

so we have

$$\psi(r) \simeq bj_0(kr) = b\frac{\sin(kr)}{kr},$$

for $r < a$.

The boundary conditions hold independently of the value of ℓ, so requiring $\psi(r=a)$ be continuous gives

$$A\left[\frac{\sin(kr)}{kr} + \frac{a_0e^{ikr}}{r}\right] = b\frac{\sin(kr)}{kr}.$$

The discontinuity in $\psi'(r=a)$ yields

$$-\frac{\hbar^2}{2m}\int \frac{\mathrm{d}^2 u}{\mathrm{d}r^2}\mathrm{d}r + \int \left[\alpha\delta(r-a) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right] u\mathrm{d}r = E\int u\mathrm{d}r,$$

or

$$-\frac{\hbar^2}{2m}\Delta u' + \alpha u(a) = 0.$$

Thus

$$\Delta u' = \frac{2m\alpha}{\hbar^2}u(a).$$

Now $u = rR$, so $u' = R + rR'$ and

$$\Delta u = \Delta R + a\Delta R' = a\Delta R' = \frac{2m\alpha}{\hbar^2}aR(a).$$

This means

$$\Delta\psi' = \frac{2m\alpha}{\hbar^2}\psi(a) = \frac{\beta}{a}\psi(a),$$

or

$$\frac{A}{ka}\left[k\cos(ka) + a_0 ik^2 e^{ika}\right] - \frac{A}{ka^2}\left[k\sin(ka) + a_0 k e^{ika}\right] - \frac{b}{ka}k\cos(ka) + \frac{b}{ka^2}\sin(ka) = \frac{\beta}{a}\frac{b\sin(ka)}{ka}.$$

The second and fourth terms on the left hand side cancel, because of the continuity equation for $\psi(r=a)$. Now we use that condition again, to eliminate b, to give

$$\begin{aligned} A\left[\cos(ka) + a_0 ike^{ika}\right] &= b\left[\cos(ka) + \beta\frac{\sin(ka)}{ka}\right] \\ &= \left[\cot(ka) + \frac{\beta}{ka}\right]\left[\sin(ka) + a_0 k e^{ika}\right]A. \end{aligned}$$

Multiplying out the right hand side, this becomes

$$\cos(ka) + a_0 ike^{ika} = \cos(ka) + \frac{\beta}{ka}\sin(ka) + a_0 k\cot(ka)e^{ika} + \frac{\beta a_0}{a}e^{ika},$$

or

$$ia_0 k\left[1 + i\cot(ka) + i\frac{\beta}{ka}\right] = \frac{\beta}{ka}\sin(ka).$$

Now we can expand for $ka \ll 1$, so $\sin(ka) \simeq ka$ and $\cot(ka) \simeq (ka)^{-1}$. This leads us to

$$ia_0 k(1 + ika)\left[1 + \frac{i}{ka}(1+\beta)\right] = \beta,$$

or

$$ia_0 k\left[1+\frac{i}{ka}(1+\beta)+ika-1-\beta\right]=\beta.$$

The left hand side of this is

$$ia_0 k\left[1+\frac{i}{ka}(1+\beta)+ika-1-\beta\right]\simeq ia_0 k\left[\frac{i}{ka}(1+\beta)\right].$$

Putting these together, we deduce

$$\boxed{a_0=-\frac{a\beta}{1+\beta}}.$$

Now, Equation 10.25 is

$$f(\theta)=\sum_{\ell=0}^{\infty}(2\ell+1)a_\ell P_\ell(\cos\theta),$$

and Equation 10.14 is

$$D(\theta)=\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=|f(\theta)|^2.$$

In our case these give us

$$f(\theta)\simeq a_0=\boxed{-\frac{a\beta}{1+\beta}},$$

and

$$D=|f|^2=\boxed{\left(\frac{\alpha\beta}{1+\beta}\right)^2}.$$

Finally, we have Equation 10.27,

$$\sigma=4\pi\sum_{\ell=0}^{\infty}(2\ell+1)|a_\ell|^2$$

or

$$\sigma=4\pi D=\boxed{4\pi\left(\frac{\alpha\beta}{1+\beta}\right)^2}.$$

Question 2 [Griffiths 10.5]

A particle of mass m and energy E is incident from the left on the potential

$$V(x)=\begin{cases}0 & x<-a,\\ -V_0 & -a\le x\le 0,\\ \infty & x>0.\end{cases}$$

(a) If the incoming wave is Ae^{ikx} (where $k = \sqrt{2mE}/\hbar$), find the reflected wave. *Answer*:

$$Ae^{-2ika}\left[\frac{k - ik'\cot(k'a)}{k + ik'\cot(k'a)}\right], \qquad \text{where} \qquad k' = \sqrt{2m(E+V_0)}/\hbar.$$

(b) Confirm that the reflected wave has the same amplitude as the incident wave.

(c) Find the phase shift δ [defined through Equation 10.40],

$$\psi(x) = A\left(e^{ikx} - e^{i(2\delta - kx)}\right),$$

for a very deep well ($E \ll V_0$). *Answer*: $\delta = -ka$.

Solution 2

(a) In the region to the left, the wavefunction is

$$\psi(x) = Ae^{ikx} + Be^{-ikx},$$

for $x \le -a$. In the central region, with $-a \le x < 0$, the Schrödinger equation gives us

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) - V_0\psi(x) = E\psi(x),$$

which we can write as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = -(k')^2\psi(x),$$

with $k' = \sqrt{2m(E+V_0)}/\hbar$. The general solution in this region is

$$\psi(x) = C\sin(k'x) + D\cos(k'x).$$

The boundary condition $\psi(0) = 0$ means that $D = 0$, so

$$\psi(x) = C\sin(k'x)$$

for $-a \le x < 0$. We can now use continuity at $x = a$ to find B. The wavefunction must be continuous, so

$$Ae^{-ika} + Be^{ika} = -C\sin(k'a),$$

and the derivative must also be continuous, so we also have

$$Aike^{-ika} - ikBe^{ika} = k'C\cos(k'a).$$

There are several ways to solve this pair of equations, but one is to divide:

$$\frac{Aike^{-ika} - ikBe^{ika}}{Ae^{-ika} + Be^{ika}} = -k' \cot(k'a).$$

Rearranging this gives

$$Aike^{-ika} - ikBe^{ika} = -Ae^{-ika}k' \cot(k'a) - Be^{ika}k' \cot(k'a),$$

or

$$Be^{ika}\left[-ik + k' \cot(k'a)\right] = Ae^{-ika}\left[-ik - k' \cot(k'a)\right].$$

Thus

$$\boxed{B = Ae^{-2ika}\left[\frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)}\right]}.$$

(b) The coefficients squared are equal:

$$|B|^2 = |A|^2 e^{-2ika} \cdot e^{2ika}\left[\frac{(k - ik' \cot(k'a))}{(k + ik' \cot(k'a))}\frac{(k + ik' \cot(k'a))}{(k - ik' \cot(k'a))}\right] = |A|^2.$$

(c) We can use our result from part (a) to write the wavefunction (in the region $x \leq a$) as

$$\psi(x) = Ae^{ikx} + Ae^{-2ika}\left[\frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)}\right] e^{-ikx}.$$

The phase shift is defined by Equation 10.40

$$\psi(x) = A\left(e^{ikx} - e^{i(2\delta - kx)}\right)$$

so

$$\psi(x) = A\left[e^{ikx} - e^{i(2\delta - kx)}\right],$$

from which we deduce the exact result

$$e^{-2ika}\left[\frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)}\right] = -e^{2i\delta}.$$

In the approximation that the well is very deep, we have $E \ll V_0$, so

$$k = \frac{\sqrt{2mE}}{\hbar} \ll \frac{\sqrt{2m(E + V_0)}}{\hbar} = k'.$$

Then we can approximate our exact result as

$$e^{-2ika}\left[\frac{-ik' \cot(k'a)}{ik' \cot(k'a)}\right] = -e^{2i\delta},$$

or

$$e^{-2ika} = e^{2i\delta}.$$

Therefore

$$\boxed{\delta = -ka}.$$

Question 3 [Griffiths 10.22]

By analogy with Section 10.2, develop partial wave analysis for two dimensions.

(a) In polar coordinates (r, θ) the Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Find the separable separations solutions to the (time-independent) Schrödinger equation, for a potential with azimuthal symmetry ($V(r, \theta) \to V(r)$). *Answer*:

$$\psi(r, \theta) = R(r)e^{ij\theta},$$

where j is an integer, and $u \equiv \sqrt{r}R$ satisfies the radial equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{(j^2 - 1/4)}{r^2}\right]u = Eu.$$

(b) By solving the radial equation for very large r (where both $V(r)$ and the centrifugal term go to zero), show that an outgoing radial wave has the asymptotic form

$$R(r) \sim \frac{e^{ikr}}{\sqrt{r}},$$

where $k \equiv \sqrt{2mE}/\hbar$. Check that an incident wave of the form Ae^{ikx} satisfies the Schrödinger equation, for $V(r) = 0$ (this is trivial, if you Cartesian coordinates). Write down the two-dimensional analog to Equation 10.12, and compare your result to Problem 10.2 *Answer*:

$$\psi(r, \theta) \approx A\left[e^{ikx} + f(\theta)\frac{e^{ikr}}{\sqrt{r}}\right], \qquad \text{for large } r.$$

(c) Construct the analog to Equation 10.21 (the wavefunction in the region where $V(r) = 0$ but the centrifugal term *cannot* be ignored). *Answer*:

$$\psi(r, \theta) \approx A\left[e^{ikx} + \sum_{j=-\infty}^{\infty} c_j H_j^{(1)}(kr)e^{ij\theta}\right],$$

where $H_j^{(1)}$ is the Hankel function (*not* the *spherical* Hankel function!) of order j.

(d) For large z,

$$H_j^{(1)}(z) \sim \sqrt{\frac{2}{\pi}} e^{-i\pi/4}(-i)^l \frac{e^{iz}}{\sqrt{z}}.$$

Use this to show that

$$f(\theta) = \sqrt{\frac{2}{\pi k}} e^{-i\pi/4} \sum_{j=-\infty}^{\infty} (-i)^j c_j e^{ij\theta}.$$

(e) Adapt the argument of Section 10.1.2 to this two-dimensional geometry. Instead of the *area* σ, we have a *length*, $\mathrm{d}b$, and in place of the solid angle $\mathrm{d}\Omega$ we have the increment of scatterin angle $|\mathrm{d}\theta|$; the role of the differential cross-section is played by

$$D(\theta) \equiv \left| \frac{\mathrm{d}b}{\mathrm{d}\theta} \right|,$$

and the effective "width" of the target (analogous to the total cross-section) is

$$B \equiv \int_0^{2\pi} D(\theta)\mathrm{d}\theta.$$

Show that

$$D(\theta) = |f(\theta)|^2, \qquad \text{and} \qquad B = \frac{4}{k} \sum_{j=-\infty}^{\infty} |c_j|^2.$$

(f) Consider the case of scattering from a hard disk (or, in three dimensions, an infinite cylinder) of radius a:

$$V(r) = \begin{cases} \infty, & (r \leq a), \\ 0, & (r > a). \end{cases}$$

By imposting appropriate boundary conditions at $r = a$, determine B. You'll need the analog to Rayleigh's formula:

$$e^{ikx} = \sum_{j=-\infty}^{\infty} i^j J_j(kr) e^{ij\theta}$$

(where J_j is the Bessel function of order J). Plot B as a function of ka, for $0 < ka < 2$.

Solution 3

Start by plugging the Ansatz into the Schrödinger equation, write $\psi(r, \theta) = R(r)\Theta(\theta)$, then multiply by r^2 and divide by $R(r)\Theta(\theta)$. This will lead to...