Quantum Mechanics II: PHYS 314 (Spring 2021)
Midterm 2–Solutions.

Overview

In this midterm you will study the free electron gas model of pure iron, and then apply your knowledge of quantum mechanical scattering to identical particles.

There are **two questions**, for a total of **100 points**. **Answer both questions**. You are welcome to use the textbook and other internet resources, but you should cite your sources appropriately. You may ask clarifying questions via slack, but you cannot discuss the midterm with other students. During this midterm, you are expected to abide by the Honor Code. Please return your solutions to me as pdf files, with your name in the title of the file, by 09:29 am on **Thursday April 1**.

Question 1 — 40pts

(a) Assuming that we can model iron as a free electron gas, calculate the Fermi energy (in electron volts) and degeneracy pressure for iron. The free electron density of iron is 17.0 $\times 10^{28}$ m^{-3}.

(b) Set the (nonrelativistic) kinetic energy of the free electrons to be equal to their Fermi energy, and calculate the "Fermi speed", v_F, of the electrons.

(c) How does your answer change for the Fermi speed change if you use the relativistic kinetic energy $E = (\gamma - 1)mc^2$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$?

(d) Are these electrons nonrelativistic? How could you have estimated this without explicitly calculating the velocity of the free electrons?

(e) Calculate the Fermi temperature of iron, which is the temperature at which the characteristic thermal energy equals the Fermi energy. The thermal energy is given by $k_B T$, where k_B is the Boltzmann constant and T is temperature in Kelvin. At the Fermi temperature the metal is considered "hot", because thermal effects become important when understanding conductivity in metals. The melting point of iron is around 1500°C. Are thermal effects important corrections to the free electron gas model of solid iron?

Solution 1

(a) 10 points

The energy is given by

$$
\begin{aligned}
E_F &= \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \\
&= \frac{(1.055\times10^{-34}\,\mathrm{J}\cdot s)(6.58\times10^{-16}\,\mathrm{eV}\cdot\mathrm{s})}{2\cdot(9.109\times10^{-31}\,\mathrm{kg})}(3\pi^2\cdot17.0\times10^{28}\,\mathrm{m}^{-3})^{2/3} \\
&= \boxed{11.2\,\mathrm{eV}}
\end{aligned}
$$

Note that the correct number of significant matches the least precise quantity in the question, which is the free electron density.

The degeneracy pressure is

$$
\begin{aligned}
P &= \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \\
&= \frac{(3\pi^2)^{3/2}(1.055\times10^{-34}\,\mathrm{J}\cdot\mathrm{s})^2}{5(9.109\times10^{-31}\,\mathrm{kg})}(8.49\times10^{28}\,\mathrm{m}^{-3})^{5/3} \\
&= \boxed{1.22\times10^{11}\,\mathrm{N/m}^2}.
\end{aligned}
$$

(b) 10 points

We equate the Fermi energy to $mv^2/2$, which gives

$$
11.19\,\mathrm{eV} = \frac{1}{2}(0.511\times10^6\,\mathrm{eV}/c^2)v^2,
$$

where we've expressed the mass of the electron in units of eV/c^2. Then we have

$$
\frac{v^2}{c^2} = \frac{22.38}{0.511\times10^6} = 4.38\times10^{-5},
$$

or

$$
\frac{v}{c} = 6.61\times10^{-3}.
$$

Thus the speed is

$$
v = (6.61\times10^{-3})\cdot(3.00\times10^8\,\mathrm{m/s}) = \boxed{1.98\times10^6\,\mathrm{m/s}}.
$$

(c) 10 points

In this case we equate the Fermi energy to $(\gamma-1)mc^2$, which gives

$$
11.19\,\mathrm{eV} = \left[\frac{1}{\sqrt{1-v^2/c^2}} - 1\right](0.511\times10^6\,\mathrm{eV}/c^2)c^2.
$$

This leads to

$$\frac{1}{1+\frac{11.19}{0.511\times 10^6}} = \sqrt{1-v^2/c^2},$$

which we rearrange as

$$\frac{v^2}{c^2} = 1 - \left[\frac{1}{1+\frac{11.19}{0.511\times 10^6}}\right]^2,$$

so that

$$\frac{v}{c} = \sqrt{1 - \left[\frac{1}{1+\frac{11.19}{0.511\times 10^6}}\right]^2} = 6.62\times 10^{-3}.$$

In other words,

$$v = (6.62\times 10^{-3})\cdot(3.00\times 10^8\,\mathrm{m/s}) = \boxed{1.99\times 10^6\,\mathrm{m/s}}.$$

(d) 5 points

We see that the relativistic speed is almost identical to the nonrelativistic speed, which tells us that relativistic corrections are very small. In fact, they are approximately

$$\frac{v_{\mathrm{rel.}} - v_{\mathrm{NR}}}{v_{\mathrm{NR}}} \times 100\% = 0.15\%.$$

Therefore the electrons are **nonrelativistic**.

How could we have estimated this, without calculating the energy? The key is that we can look at the relative size of the Fermi energy in electron volts and the mass of the electron. To compare these numbers properly, we need to compare E_F/c^2 with m_e, expressed in eV/c^2. An order of magnitude estimate gives

$$\frac{E_F}{c^2} \sim 10\,\mathrm{eV/c^2} \ll m_e = 0.511\times 10^6\,\mathrm{eV/c^2}.$$

The **kinetic energy of the electrons is much smaller than the rest mass of the electron** and therefore the system is nonrelativistic.

(e) 5 points

The Fermi temperature is given by

$$T = \frac{11.19\,\mathrm{eV}}{8.62\times 10^{-5}\,\mathrm{eV/K}} = \boxed{1.30\times 10^5\,\mathrm{K}}.$$

This is significantly higher than the melting point of iron, which is around 1.8×10^3 K, so thermal effects are (likely) **not siginificant corrections to the free electron gas model of solid iron**.

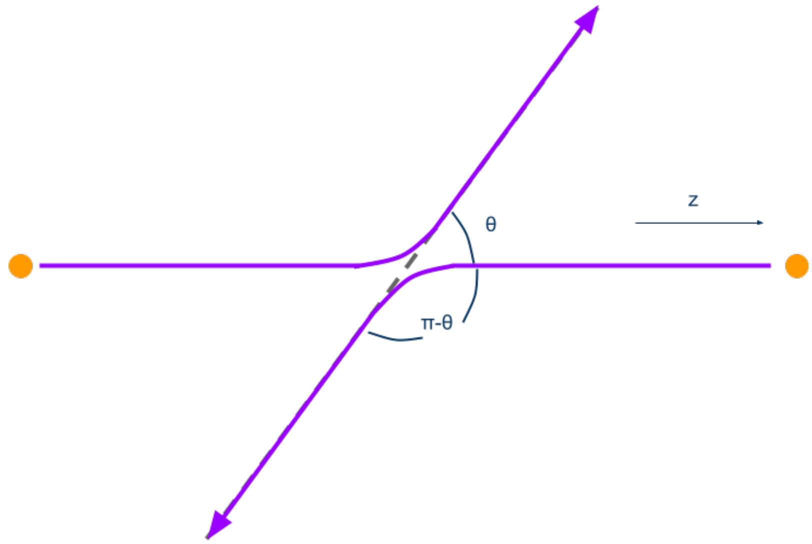

Figure 1: Scattering two particles in the centre of mass frame.

Question 2 **60pts**

Consider the scattering of two identical particles from a central potential in the centre of mass frame, illustrated in figure 1. Assume that the incident particles are incident in the positive and negative z-directions. For classical scattering, the total cross-section is given by

$$\sigma_{\text{cl.}}(\theta) = \sigma(\theta) + \sigma(\pi - \theta).$$

In quantum mechanics, we must account for the differences that arise when we have incident bosons or incident fermions. We can tackle this situation by introducing the centre of mass coordinate $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and the relative coordinate $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, and writing the (time-independent) wavefunction as

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{R}, \mathbf{r}) = \psi_R(\mathbf{R})\psi_r(\mathbf{r}).$$

(a) Show that $\psi_r(\mathbf{r})$ must be an even function of $\mathbf{r}$ when the incident particles are identical bosons.

(b) Construct a symmetric wavefunction that generalizes the one particle scattering wavefunction

$$\psi(r, \theta) \approx A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right], \tag{1}$$

where $r = |\mathbf{r}|$ to the case of two identical incident bosons. You may express your result in terms of an as-yet unspecified symmetric scattering amplitude $f_{\text{sym.}}(\theta)$.

Hints: Use the fact that, for a central potential, if $\psi_r(\mathbf{r})$ is a solution of the time-independent Schrödinger equation with energy E, then so is $\psi_r(-\mathbf{r})$. You will also need to consider how the reflection $\mathbf{r} \to -\mathbf{r}$ is implemented in polar coordinates (r, θ, ϕ).

(c) Find the explicit form of the symmetric amplitude (in terms of the single particle scattering amplitude that appears in the one-particle wavefunction in equation (1)).

Use this to show that the quantum scattering of identical bosons leads to an *interference term* in the differential cross-section

$$2\,\mathrm{Re}\left[f^*(\theta)f(\pi-\theta)\right]$$

that is not present in classical scattering of two particles.

(d) Consider now the case of two incident identical fermions in the spin triplet state. Show that the scattering amplitude for this case vanishes at $\theta = \pi/2$ and find the differential cross-section (for arbitrary scattering angle θ).

(e) What is the differential cross-section for incident identical fermions in the spin singlet state?

Solution 2

(a) 5 points
Under interchange of the two particles, the relative coordinate changes sign, $\mathbf{r} \to -\mathbf{r}$, but the centre of mass coordinate is unchanged. The whole wavefunction must be symmetric, so we must have $\psi_r(-\mathbf{r}) = \psi_r(\mathbf{r})$. In other words, the wavefunction is an even function of $\mathbf{r}$.

(b) 15 points
Both $\psi_r(\mathbf{r})$ and $\psi_r(-\mathbf{r})$ are solutions of the time-independent Schrödinger equation with the same energy. Therefore we can construct the general solution as a symmetric linear combination of these two solutions

$$\boxed{\psi_{\mathbf{sym.}}(\mathbf{r}) = A\left[e^{ikz} + e{-}ikz f_{\mathrm{sym.}}\frac{e^{ikr}}{r}\right]}.$$

Note that in polar coordinates, $\mathbf{r} \to -\mathbf{r}$ is implemented through $(r, \theta, \phi) \to (r, \pi - \theta, \pi + \phi)$. Therefore, the outgoing radial wave, which is only a function of r, is unchanged under interchange of the particles.

(c) 15 points
Using the same arguments for the relative coordinate (in polar coordinates) under

particle interchange, we have that

$$f_{\text{sym.}} = f_1(\theta) + f_2(\theta)$$

must be symmetric under $1 \leftrightarrow 2$, which also corresponds to $\theta \leftrightarrow \pi - \theta$. Therefore we deduce that

$$\boxed{f_{\text{sym.}} = f(\theta) + f(\pi - \theta)}$$

The differential cross-section is given by

$$\begin{aligned}
\frac{\mathrm{d}\sigma_{\text{sym.}}}{\mathrm{d}\Omega} &= |f(\theta) + f(\pi - \theta)|^2 \\
&= |f(\theta)|^2 + |f(\pi - \theta)|^2 + f^*(\theta) f(\pi - \theta) + f(\theta) f^*(\pi - \theta) \\
&= |f(\theta)|^2 + |f(\pi - \theta)| + 2 \operatorname{Re}\left[f^*(\theta) f(\pi - \theta)\right],
\end{aligned}$$

so that the interference term is

$$2 \operatorname{Re}\left[f^*(\theta) f(\pi - \theta)\right],$$

as required.

(d) 20 points

Two identical fermions in a spin triplet state have a symmetric spin wavefunction, so their spatial wavefunction must be antisymmetric. This means that the wavefunction must satisfy

$$\psi_r(-\mathbf{r}) = -\psi_r(\mathbf{r}).$$

Applying the same reasoning as we used for the bosonic case, this means that the scattering amplitude must be

$$f_{\text{triplet}} = f(\theta) - f(\pi - \theta).$$

At $\theta = \pi/2$, this becomes

$$f_{\text{triplet}} = f(\pi/2) - f(\pi - \pi/2) = f(\pi/2) - f(\pi/2) = \boxed{0},$$

as required.

The differential cross-section becomes

$$\begin{aligned}
\frac{\mathrm{d}\sigma_{\text{triplet}}}{\mathrm{d}\Omega} &= |f(\theta) - f(\pi - \theta)|^2 \\
&= |f(\theta)|^2 + |f(\pi - \theta)|^2 - f^*(\theta) f(\pi - \theta) - f(\theta) f^*(\pi - \theta) \\
&= \boxed{|f(\theta)|^2 + |f(\pi - \theta)| - 2 \operatorname{Re}\left[f^*(\theta) f(\pi - \theta)\right]}.
\end{aligned}$$

(e) 5 points

Two identical fermions in a spin singlet state have antisymmetric spin wavefunction, so their spatial wavefunction must besymmetric. This means that the wavefunction must satisfy

$$\psi_r(-\mathbf{r}) = \psi_r(\mathbf{r}),$$

as in the case of two identical bosons. Therefore the differential cross-section is

$$\boxed{\frac{\mathrm{d}\sigma_{\text{singlet}}}{\mathrm{d}\Omega} = |f(\theta)|^2 + |f(\pi-\theta)| + 2\,\mathrm{Re}\left[f^*(\theta)f(\pi-\theta)\right]}.$$