Quantum Mechanics II: PHYS 314 (Spring 2021)
Midterm 1–Solutions.

Overview

In this midterm you will apply your understanding of symmetries in quantum mechanics to two new transformations, time reversal and charge conjugation. These transformations have some rather interesting properties: time reversal is an anti-linear operator and charge conjugation acts on the states themselves, rather than on spatial coordinates. The first two questions will explore these properties in a little more detail. In the third question, you will apply your knowledge of multiparticle systems to study the properties of a three-boson state. The background information is for your interest and to provide context to the questions. A full understanding of the background discussion is not necessary to answer the questions.

There are **three questions**, for a total of **100 points**. **Answer all three questions**. You are welcome to use the textbook and other internet resources, but you should cite your sources appropriately. You may ask clarifying questions via slack, but you cannot discuss the midterm with other students. During this midterm, you are expected to abide by the Honor Code. Please return your solutions to me as pdf files, with your name in the title of the file, by 11:59 pm on **Wednesday March 3**.

Question 1 — 40pts

Background In contrast to the operators that we have seen before, the time-reversal operator is **anti-linear**. One of the tricky things about anti-linear operators is that they make Dirac notation a little ambiguous. For linear operators, we can always write an expectation value of some operator, $\mathcal{O}$, as $\langle\psi|\widehat{\mathcal{O}}|\psi\rangle$, because it doesn't matter whether the operator acts to the left or to the right. Remember that bra-states (or dual states) are really linear operators that map states (vectors) to real numbers. This means that the dual state $\langle\psi|\widehat{\mathcal{O}}$ is defined through

$$\Big[\langle\psi|\widehat{\mathcal{O}}\Big]|\psi\rangle = \langle\psi|\Big[\widehat{\mathcal{O}}|\psi\rangle\Big].$$

Unfortunately, for anti-linear operators, things are not so simple. If we define the dual state of an anti-linear operator in the same way as we did for linear operators, then it turns out the dual state is anti-linear too (and not linear, as it should be). The correct definition is actually

$$\Big[\langle\psi|\widehat{\mathcal{A}}\Big]|\psi\rangle = \Big(\langle\psi|\Big[\widehat{\mathcal{A}}|\psi\rangle\Big]\Big)^*,$$

for an anti-linear operator $\widehat{\mathcal{A}}$. Therefore we should never use the notation $\langle\psi|\widehat{\mathcal{A}}|\psi\rangle$ for an anti-linear operator, because it is ambiguous (the result depends on whether the operator acts to the left or the right).

Questions The time-reversal operator satisfies the anti-linear property

$$\widehat{\mathcal{T}}\left[\alpha|\psi_1\rangle + \beta|\psi_2\rangle\right] = \alpha^*\widehat{\mathcal{T}}|\psi_1\rangle + \beta^*\widehat{\mathcal{T}}|\psi_2\rangle,$$

where α and β are complex numbers, and the star indicates complex conjugation. When acting on operators, time reversal acts to switch the momentum of a particle acting on a time-reversed test function, but has no effect on the spatial axes, so it satisfies

$$\mathcal{T}\widehat{\mathbf{p}}\widehat{\mathcal{T}}^{-1} = -\widehat{\mathbf{p}}, \qquad \text{and} \qquad \widehat{\mathcal{T}}\widehat{\mathbf{x}}\widehat{\mathcal{T}}^{-1} = \widehat{\mathbf{x}}.$$

(a) Show, by considering the commutator of time-reversed operators, that the time reversal operator **must** be anti-linear for consistency with the commutator of the position and momentum operators

$$[\widehat{x}_i, \widehat{p}_j] = i\hbar\delta_{ij}.$$

(b) By considering a general state in the position basis

$$|\psi\rangle = \int \mathrm{d}^3x\,|\mathbf{x}\rangle\langle\mathbf{x}|\psi\rangle = \int \mathrm{d}^3x\,\psi(\mathbf{x})|\mathbf{x}\rangle,$$

show that the time-reversal operator acts on wavefunctions as

$$\widehat{\mathcal{T}}\psi(\mathbf{x}) = \psi^*(\mathbf{x}).$$

(c) Using this result, check that the momentum eigenstate (in the position basis)

$$|\mathbf{p}\rangle = \int \mathrm{d}^3\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}}|\mathbf{x}\rangle$$

reverses its momentum, as expected.

(d) Time-reversal invariance corresponds to time-evolving a state, reversing its momentum, then time-evolving the state again for the same amount of time. Mathematically, this means time-reversal invariance corresponds to

$$\widehat{U}(t)\widehat{\mathcal{T}}\widehat{U}(t) = \widehat{\mathcal{T}}.$$

By considering an infinitesimal time interval, show that time-reversal invariance requires

$$\left[\widehat{\mathcal{T}}, \widehat{H}\right] = 0.$$

(e) Show that, for a time-reversal invariant Hamiltonian, if $\psi_n(\mathbf{x})$ is a stationary state with energy E_n, then $\psi_n^*(\mathbf{x})$ is also a stationary state with the same energy.

Question 2 **40pts**

Background The charge conjugation operator $\widehat{\mathcal{C}}$ is another tricky customer, because it acts not on coordinates, but on states themselves. In relativistic quantum theories, the charge conjugation operator transforms particles into antiparticles, and vice versa. Here we will model this process with a two state system, and we will not identify the states as "particles" or "antiparticles"–we will simply just treat them as two separate states, of opposite charge. This is a toy model of the phenenomen of "neutral meson mixing", which occurs in the kaon, D-meson and B-meson systems (mesons are combinations of pairs of quarks, which are the constituents that make up protons, neutrons and many other subatomic particles).

Charge conjugation, along with parity and time reversal (which you have already seen) are together called "CPT" and invariance under simultaneous CPT operations is a fundamental property of our Universe at the smallest scales (as far as we can tell so far, anyway). If you find a relativistic quantum system where, say, neither charge conjugation nor parity are symmetries of the theory (for example, the weak nuclear force breaks both charge and parity invariance at the same time), then you can be sure that time reversal invariance must also be broken, so that the combined operation of charge conjugation, parity, and time reversal **is** a symmetry of the theory.

Questions Consider a two-state system with states $|\phi\rangle$ and $|\overline{\phi}\rangle$ that satisfy the Schrödinger equations

$$i\hbar\frac{\partial}{\partial t}|\phi\rangle = \widehat{H}|\phi\rangle, \qquad \text{and} \qquad i\hbar\frac{\partial}{\partial t}|\overline{\phi}\rangle = \widehat{H}|\overline{\phi}\rangle,$$

and are orthonormal. These two states have opposite charge of $q = \pm 1$, which we can represent through the "charge operator", $\widehat{\mathcal{Q}}$, acting on the states as

$$\widehat{\mathcal{Q}}|\phi\rangle = |\phi\rangle, \qquad \text{and} \qquad \widehat{\mathcal{Q}}|\overline{\phi}\rangle = -|\overline{\phi}\rangle.$$

Under charge conjugation, these states are transformed into (the negative of) each other

$$\widehat{\mathcal{C}}|\phi\rangle = -|\overline{\phi}\rangle, \qquad \text{and} \qquad \widehat{\mathcal{C}}|\overline{\phi}\rangle = -|\phi\rangle.$$

(a) Show that the charge conjugation operator has eigenvalues of ± 1.

(b) Show that a general eigenstate of the charge operator with charge q,

$$\widehat{\mathcal{Q}}|q\rangle = q|q\rangle,$$

can only be an eigenstate of the charge conjugation operator if $q = 0$. In other words, only neutral particles are eigenstates of the charge conjugation operator!

(c) Assuming that both $|\phi\rangle$ and $|\overline{\phi}\rangle$ are **odd** under parity transformations, find the linear combinations of $|\phi\rangle$ and $|\overline{\phi}\rangle$ that are eigenstates of the combined operation of charge conjugation and parity, $\widehat{\mathcal{C}}\widehat{P}$. Show that their eigenvalues match expectations from part (a) of this question.

(d) Do you expect there to be a conserved quantity associated with symmetry under charge conjugation? Explain your reasoning.

Question 3 — 20pts

Three identical spin-one bosons are in a symmetric spatial state, described by the wavefunction $\phi(\mathbf{r}_1)\phi(\mathbf{r}_3)\phi(\mathbf{r}_3)$. Write down all possible normalised spin wavefunctions of the system and the corresponding values of the total spin, projected onto the z-axis.

Solution 1

(a) 10 points

Let us consider the commutator acting on a test function $f(\mathbf{x})$,

$$[\widehat{x}_i, \widehat{p}_j] f(\mathbf{x}) = \widehat{x}_i \widehat{p}_j f(\mathbf{x}) - \widehat{p}_j \widehat{x}_i f(\mathbf{x}) = i\hbar\delta_{ij} f(\mathbf{x}).$$

We left-multiply this by the time-reversal operator, and then insert the identity in the form $\widehat{\mathcal{T}}^{-1}\widehat{\mathcal{T}} = \mathbb{I}$ to give

$$\widehat{\mathcal{T}}\widehat{x}_i\widehat{\mathcal{T}}^{-1}\widehat{\mathcal{T}}\widehat{p}_j\widehat{\mathcal{T}}^{-1}\widehat{\mathcal{T}} f(\mathbf{x}) - \widehat{\mathcal{T}}\widehat{p}_j\widehat{\mathcal{T}}^{-1}\widehat{\mathcal{T}}\widehat{x}_i\widehat{\mathcal{T}}^{-1}\widehat{\mathcal{T}} f(\mathbf{x}) = \widehat{\mathcal{T}}\Big[i\hbar\delta_{ij} f(\mathbf{x})\Big]. \tag{1}$$

On the left hand side, we apply the operator transformation rules given in the question to obtain

$$\widehat{x}_i\Big(-\widehat{p}_j\Big)\widehat{\mathcal{T}} f(\mathbf{x}) - \Big(-\widehat{p}_j\Big)\widehat{x}_i\widehat{\mathcal{T}} f(\mathbf{x}) = -[\widehat{x}_i, \widehat{p}_j]\widehat{\mathcal{T}} f(\mathbf{x}) = -i\hbar\delta_{ij}\widehat{\mathcal{T}} f(\mathbf{x}).$$

For this to be consistent with equation (1), the time-reversal operator **has to be anti-linear**, in that it must satisfy

$$\widehat{\mathcal{T}}\Big[i\hbar\delta_{ij} f(\mathbf{x})\Big] = (i\hbar\delta_{ij})^* \,\widehat{\mathcal{T}} f(\mathbf{x}) = -i\hbar\delta_{ij}\widehat{\mathcal{T}} f(\mathbf{x}),$$

as required.

Note: one can also consider the commutator

$$[\widehat{x}_i', \widehat{p}_j'] = \widehat{\mathcal{T}} i\hbar\delta_{ij}\widehat{\mathcal{T}}^{-1},$$

where the prime indicates that these are time-reversed operators.

(b) 6 points

Acting on a general state in the position basis with the time-reversal operator gives

$$\widehat{\mathcal{T}}|\psi\rangle = \int \mathrm{d}^3x\, \widehat{\mathcal{T}}\Big[\psi(\mathbf{x})|\mathbf{x}\rangle\Big] = \int \mathrm{d}^3x\, \psi^*(\mathbf{x})\widehat{\mathcal{T}}|\mathbf{x}\rangle = \int \mathrm{d}^3x\, \psi^*(\mathbf{x})|\mathbf{x}\rangle.$$

Therefore we deduce that

$$\widehat{\mathcal{T}}\psi(\mathbf{x}) = \psi^*(\mathbf{x}),$$

as required.

(c) 4 points

We apply the time-reversal operator to a momentum state in the position basis to obtain

$$\widehat{\mathcal{T}}|\mathbf{p}\rangle = \int \mathrm{d}^3\mathbf{x}\, \widehat{\mathcal{T}}\left[e^{i\mathbf{p}\cdot\mathbf{x}}|\mathbf{x}\rangle\right] = \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\widehat{\mathcal{T}}|\mathbf{x}\rangle = \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{x}\rangle = |-\mathbf{p}\rangle,$$

as required.

(d) 10 points

Let the evolved time be $t = \delta$, which we take to be infinitesimal. Then the lefthand side of our equation

$$\widehat{U}(t)\widehat{\mathcal{T}}\widehat{U}(t) = \widehat{\mathcal{T}},$$

becomes

$$\begin{aligned}
\exp\left[-\frac{i\delta}{\hbar}\widehat{H}\right]\widehat{\mathcal{T}}\exp\left[-\frac{i\delta}{\hbar}\widehat{H}\right] &\approx \left[1-\frac{i\delta}{\hbar}\widehat{H}\right]\widehat{\mathcal{T}}\left[1-\frac{i\delta}{\hbar}\widehat{H}\right] \\
&= \widehat{\mathcal{T}} - \frac{i\delta}{\hbar}\widehat{H}\widehat{\mathcal{T}} - \widehat{\mathcal{T}}\left[\frac{i\delta}{\hbar}\widehat{H}\right] \\
&= \widehat{\mathcal{T}} - \frac{i\delta}{\hbar}\widehat{H}\widehat{\mathcal{T}} - \left[\frac{i\delta}{\hbar}\right]^*\widehat{\mathcal{T}}\widehat{H} \\
&= \widehat{\mathcal{T}} - \frac{i\delta}{\hbar}\widehat{H}\widehat{\mathcal{T}} + \frac{i\delta}{\hbar}\widehat{\mathcal{T}}\widehat{H} \\
&= \widehat{\mathcal{T}} + \frac{i\delta}{\hbar}\left[\widehat{\mathcal{T}}, \widehat{H}\widehat{\mathcal{T}}\right].
\end{aligned}$$

For this lefthand side to equal the righthand side, $\widehat{\mathcal{T}}$, we must have

$$\left[\widehat{\mathcal{T}}, \widehat{H}\right] = 0,$$

as required.

Note: This question requires care with the anti-linearity of $\widehat{\mathcal{T}}$, in lines two to four, to get the minus signs correct.

(e) 10 points

Let's consider the Hamiltonian acting on the complex conjugate of the wavefunction,

$$\begin{aligned}\widehat{H}\psi_n^*(\mathbf{x}) &= \widehat{H}\widehat{\mathcal{T}}\psi_n(\mathbf{x}) \\ &= \widehat{\mathcal{T}}\widehat{H}\psi_n(\mathbf{x}) \\ &= \widehat{\mathcal{T}}E_n\psi_n(\mathbf{x}) \\ &= E_n\widehat{\mathcal{T}}\psi_n(\mathbf{x}) \\ &= E_n\psi_n^*(\mathbf{x}).\end{aligned}$$

This demonstrates that $\psi_n^*(\mathbf{x})$ is an eigenstate of the Hamiltonian with the same energy, E_n, as $\psi_n(\mathbf{x})$.

Note: In the second line, we used the fact that the commutator of the two operators is zero; in the fourth line, we used the fact that the energy is real.

Solution 2

(a) 10 points

If we apply the charge conjugation operator to an eigenstate (with eigenvalue c_ψ) twice, we find

$$\widehat{\mathcal{C}}^2|\psi\rangle = \widehat{\mathcal{C}}c_\psi|\psi\rangle = c_\psi^2|\psi\rangle.$$

But we know that applying this operator twice is the same as applying the identity operator, so we must have $c_\psi^2 = 1$. Therefore we deduce that

$$\boxed{c_\psi = \pm 1},$$

as required.

(b) 15 points

The state $|q\rangle$ can only be a simultaneous eigenstate of the charge operator and the charge conjugation operator if they commute. In that case, we must have

$$\widehat{\mathcal{C}}\,\widehat{\mathcal{Q}}|q\rangle = \widehat{\mathcal{Q}}\,\widehat{\mathcal{C}}|q\rangle.$$

Let us now apply the charge conjugation operator to the state $\widehat{\mathcal{Q}}|q\rangle$,

$$\widehat{\mathcal{C}}\,\widehat{\mathcal{Q}}|q\rangle = \widehat{\mathcal{C}}q|q\rangle = q\widehat{\mathcal{C}}|q\rangle = q|-q\rangle,$$

and then consider the opposite ordering

$$\widehat{\mathcal{Q}}\,\widehat{\mathcal{C}}|q\rangle = \widehat{\mathcal{Q}}|-q\rangle = -q|-q\rangle = -q|-q\rangle.$$

Thus, if these operators commute, we must have

$$q|-q\rangle = -q|-q\rangle,$$

which means that $q = 0$, as required.

(c) 10 points

The linear combinations that are eigenstates of charge conjugation and parity are

$$\boxed{|\phi_\pm\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle \pm |\overline{\phi}\rangle\right]}.$$

These satisfy

$$\widehat{\mathcal{C}}\widehat{\mathcal{P}}|\phi_\pm\rangle = \pm|\phi_\pm\rangle,$$

so their eigenvalues are $\boxed{\pm 1}$, as expected from part (a).

(d) 5 points

No. We do not expect there to be a conserved quantity associated with charge conjugation invariance, because it is a discrete transformation, and Noether's theorem only applies to continuous symmetries.

Solution 3

This is a system of spin-one bosons, so the total wavefunction must be symmetric under particle interchange. The spatial component is $\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\phi(\mathbf{r}_3)$ and is symmetric, so the spin states must also be symmetric. Each boson carries spin one, so has possible spin states $m_s = \{-1, 0, 1\}$, which we can label $|-\rangle$, $|0\rangle$, and $|+\rangle$.

Therefore, there are ten possible wavefunctions, and they are given by

wavefunction	S_z^{tot}
$\|{+}{+}{+}\rangle,$	3
$\frac{1}{\sqrt{3}}\left[\|{+}{+}0\rangle + \|{+}{+}0\rangle + \|0{+}{+}\rangle\right],$	2
$\frac{1}{\sqrt{3}}\left[\|0\,0\,{+}\rangle + \|0\,{+}\,0\rangle + \|{+}\,0\,0\rangle\right],$	1
$\frac{1}{\sqrt{3}}\left[\|{+}{+}{-}\rangle + \|{+}{-}{+}\rangle + \|{-}{+}{+}\rangle\right],$	1
$\|000\rangle,$	0
$\frac{1}{\sqrt{6}}\left[\|{+}{-}0\rangle + \|{+}0{-}\rangle + \|0{+}{-}\rangle + \|0{-}{+}\rangle + \|{-}\,0\,{+}\rangle + \|{-}{+}0\rangle\right],$	0
$\frac{1}{\sqrt{3}}\left[\|{-}\,0\,0\rangle + \|{-}\,0\,0\rangle + \|{-}\,0\,0\rangle\right],$	−1
$\frac{1}{\sqrt{3}}\left[\|{+}{-}{-}\rangle + \|{-}{+}{-}\rangle + \|{-}{-}{+}\rangle\right],$	−1
$\frac{1}{\sqrt{3}}\left[\|{-}{-}0\rangle + \|{-}{-}0\rangle + \|0{-}{-}\rangle\right],$	−2
$\|{-}{-}{-}\rangle,$	−3