Quantum Mechanics II: PHYS 314 (Spring 2021)
Midterm 2–Due Thursday, April 1.

Overview

In this midterm you will study the free electron gas model of pure iron, and then apply your knowledge of quantum mechanical scattering to identical particles.

There are **two questions**, for a total of **100 points**. **Answer both questions**. You are welcome to use the textbook and other internet resources, but you should cite your sources appropriately. You may ask clarifying questions via slack, but you cannot discuss the midterm with other students. During this midterm, you are expected to abide by the Honor Code. Please return your solutions to me as pdf files, with your name in the title of the file, by 09:29 am on **Thursday April 1**.

Question 1 — 40pts

(a) Assuming that we can model iron as a free electron gas, calculate the Fermi energy (in electron volts) and degeneracy pressure for iron. The free electron density of iron is 17.0×10^{28} m^{-3}.

(b) Set the (nonrelativistic) kinetic energy of the free electrons to be equal to their Fermi energy, and calculate the "Fermi speed", v_F, of the electrons.

(c) How does your answer change for the Fermi speed change if you use the relativistic kinetic energy $E = (\gamma - 1)mc^2$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$?

(d) Are these electrons nonrelativistic? How could you have estimated this without explicitly calculating the velocity of the free electrons?

(e) Calculate the Fermi temperature of iron, which is the temperature at which the characteristic thermal energy equals the Fermi energy. The thermal energy is given by $k_B T$, where k_B is the Boltzmann constant and T is temperature in Kelvin. At the Fermi temperature the metal is considered "hot", because thermal effects become important when understanding conductivity in metals. The melting point of iron is around 1500°C. Are thermal effects important corrections to the free electron gas model of solid iron?

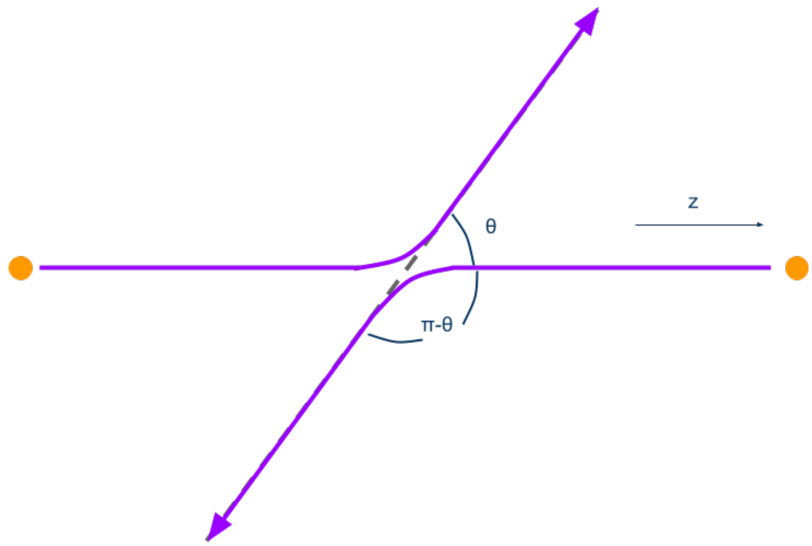

Figure 1: Scattering two particles in the centre of mass frame.

Question 2 **60pts**

Consider the scattering of two identical particles from a central potential in the centre of mass frame, illustrated in figure 1. Assume that the incident particles are incident in the positive and negative z-directions. For classical scattering, the total cross-section is given by

$$\sigma_{\text{cl.}}(\theta) = \sigma(\theta) + \sigma(\pi - \theta).$$

In quantum mechanics, we must account for the differences that arise when we have incident bosons or incident fermions. We can tackle this situation by introducing the centre of mass coordinate $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and the relative coordinate $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, and writing the (time-independent) wavefunction as

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{R}, \mathbf{r}) = \psi_R(\mathbf{R})\psi_r(\mathbf{r}).$$

(a) Show that $\psi_r(\mathbf{r})$ must be an even function of $\mathbf{r}$ when the incident particles are identical bosons.

(b) Construct a symmetric wavefunction that generalizes the one particle scattering wavefunction

$$\psi(r, \theta) \approx A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right], \tag{1}$$

where $r = |\mathbf{r}|$ to the case of two identical incident bosons. You may express your result in terms of an as-yet unspecified symmetric scattering amplitude $f_{\text{sym.}}(\theta)$.

Hints: Use the fact that, for a central potential, if $\psi_r(\mathbf{r})$ is a solution of the time-independent Schrödinger equation with energy E, then so is $\psi_r(-\mathbf{r})$. You will also need to consider how the reflection $\mathbf{r} \to -\mathbf{r}$ is implemented in polar coordinates (r, θ, ϕ).

(c) Find the explicit form of the symmetric amplitude (in terms of the single particle scattering amplitude that appears in the one-particle wavefunction in equation (1)).

Use this to show that the quantum scattering of identical bosons leads to an *interference term* in the differential cross-section

$$2\,\mathrm{Re}\left[f^*(\theta)f(\pi-\theta)\right]$$

that is not present in classical scattering of two particles.

(d) Consider now the case of two incident identical fermions in the spin triplet state. Show that the scattering amplitude for this case vanishes at $\theta = \pi/2$ and find the differential cross-section (for arbitrary scattering angle θ).

(e) What is the differential cross-section for incident identical fermions in the spin singlet state?