Quantum Mechanics II: PHYS 314 (Spring 2021)
Midterm 1–Due Wednesday, March 3.

Overview

In this midterm you will apply your understanding of symmetries in quantum mechanics to two new transformations, time reversal and charge conjugation. These transformations have some rather interesting properties: time reversal is an anti-linear operator and charge conjugation acts on the states themselves, rather than on spatial coordinates. The first two questions will explore these properties in a little more detail. In the third question, you will apply your knowledge of multiparticle systems to study the properties of a three-boson state. The background information is for your interest and to provide context to the questions. A full understanding of the background discussion is not necessary to answer the questions.

There are **three questions**, for a total of **100 points**. **Answer all three questions**. You are welcome to use the textbook and other internet resources, but you should cite your sources appropriately. You may ask clarifying questions via slack, but you cannot discuss the midterm with other students. During this midterm, you are expected to abide by the Honor Code. Please return your solutions to me as pdf files, with your name in the title of the file, by 11:59 pm on **Wednesday March 3**.

Question 1 — 40pts

Background In contrast to the operators that we have seen before, the time-reversal operator is **anti-linear**. One of the tricky things about anti-linear operators is that they make Dirac notation a little ambiguous. For linear operators, we can always write an expectation value of some operator, $\mathcal{O}$, as $\langle\psi|\widehat{\mathcal{O}}|\psi\rangle$, because it doesn't matter whether the operator acts to the left or to the right. Remember that bra-states (or dual states) are really linear operators that map states (vectors) to real numbers. This means that the dual state $\langle\psi|\widehat{\mathcal{O}}$ is defined through

$$\left[\langle\psi|\widehat{\mathcal{O}}\right]|\psi\rangle = \langle\psi|\left[\widehat{\mathcal{O}}|\psi\rangle\right].$$

Unfortunately, for anti-linear operators, things are not so simple. If we define the dual state of an anti-linear operator in the same way as we did for linear operators, then it turns out the dual state is anti-linear too (and not linear, as it should be). The correct definition is actually

$$\left[\langle\psi|\widehat{\mathcal{A}}\right]|\psi\rangle = \left(\langle\psi|\left[\widehat{\mathcal{A}}|\psi\rangle\right]\right)^*,$$

for an anti-linear operator $\widehat{\mathcal{A}}$. Therefore we should never use the notation $\langle\psi|\widehat{\mathcal{A}}|\psi\rangle$ for an anti-linear operator, because it is ambiguous (the result depends on whether the operator acts to the left or the right).

Questions The time-reversal operator satisfies the anti-linear property

$$\widehat{\mathcal{T}}\left[\alpha|\psi_1\rangle + \beta|\psi_2\rangle\right] = \alpha^*\widehat{\mathcal{T}}|\psi_1\rangle + \beta^*\widehat{\mathcal{T}}|\psi_2\rangle,$$

where α and β are complex numbers, and the star indicates complex conjugation. When acting on operators, time reversal acts to switch the momentum of a particle acting on a time-reversed test function, but has no effect on the spatial axes, so it satisfies

$$\mathcal{T}\widehat{\mathbf{p}}\widehat{\mathcal{T}}^{-1} = -\widehat{\mathbf{p}}, \qquad \text{and} \qquad \widehat{\mathcal{T}}\widehat{\mathbf{x}}\widehat{\mathcal{T}}^{-1} = \widehat{\mathbf{x}}.$$

(a) Show, by considering the commutator of time-reversed operators, that the time reversal operator **must** be anti-linear for consistency with the commutator of the position and momentum operators

$$[\widehat{x}_i, \widehat{p}_j] = i\hbar\delta_{ij}.$$

(b) By considering a general state in the position basis

$$|\psi\rangle = \int \mathrm{d}^3x\,|\mathbf{x}\rangle\langle\mathbf{x}|\psi\rangle = \int \mathrm{d}^3x\,\psi(\mathbf{x})|\mathbf{x}\rangle,$$

show that the time-reversal operator acts on wavefunctions as

$$\widehat{\mathcal{T}}\psi(\mathbf{x}) = \psi^*(\mathbf{x}).$$

(c) Using this result, check that the momentum eigenstate (in the position basis)

$$|\mathbf{p}\rangle = \int \mathrm{d}^3\mathbf{x}\,e^{i\mathbf{p}\cdot\mathbf{x}}|\mathbf{x}\rangle$$

reverses its momentum, as expected.

(d) Time-reversal invariance corresponds to time-evolving a state, reversing its momentum, then time-evolving the state again for the same amount of time. Mathematically, this means time-reversal invariance corresponds to

$$\widehat{U}(t)\widehat{\mathcal{T}}\widehat{U}(t) = \widehat{\mathcal{T}}.$$

By considering an infinitesimal time interval, show that time-reversal invariance requires

$$\left[\widehat{\mathcal{T}}, \widehat{H}\right] = 0.$$

(e) Show that, for a time-reversal invariant Hamiltonian, if $\psi_n(\mathbf{x})$ is a stationary state with energy E_n, then $\psi_n^*(\mathbf{x})$ is also a stationary state with the same energy.

Question 2 **40pts**

Background The charge conjugation operator $\widehat{\mathcal{C}}$ is another tricky customer, because it acts not on coordinates, but on states themselves. In relativistic quantum theories, the charge conjugation operator transforms particles into antiparticles, and vice versa. Here we will model this process with a two state system, and we will not identify the states as "particles" or "antiparticles"–we will simply just treat them as two separate states, of opposite charge. This is a toy model of the phenenomen of "neutral meson mixing", which occurs in the kaon, D-meson and B-meson systems (mesons are combinations of pairs of quarks, which are the constituents that make up protons, neutrons and many other subatomic particles).

Charge conjugation, along with parity and time reversal (which you have already seen) are together called "CPT" and invariance under simultaneous CPT operations is a fundamental property of our Universe at the smallest scales (as far as we can tell so far, anyway). If you find a relativistic quantum system where, say, neither charge conjugation nor parity are symmetries of the theory (for example, the weak nuclear force breaks both charge and parity invariance at the same time), then you can be sure that time reversal invariance must also be broken, so that the combined operation of charge conjugation, parity, and time reversal **is** a symmetry of the theory.

Questions Consider a two-state system with states $|\phi\rangle$ and $|\overline{\phi}\rangle$ that satisfy the Schrödinger equations

$$i\hbar\frac{\partial}{\partial t}|\phi\rangle = \widehat{H}|\phi\rangle, \qquad \text{and} \qquad i\hbar\frac{\partial}{\partial t}|\overline{\phi}\rangle = \widehat{H}|\overline{\phi}\rangle,$$

and are orthonormal. These two states have opposite charge of $q = \pm 1$, which we can represent through the "charge operator", $\widehat{\mathcal{Q}}$, acting on the states as

$$\widehat{\mathcal{Q}}|\phi\rangle = |\phi\rangle, \qquad \text{and} \qquad \widehat{\mathcal{Q}}|\overline{\phi}\rangle = -|\overline{\phi}\rangle.$$

Under charge conjugation, these states are transformed into (the negative of) each other

$$\widehat{\mathcal{C}}|\phi\rangle = -|\overline{\phi}\rangle, \qquad \text{and} \qquad \widehat{\mathcal{C}}|\overline{\phi}\rangle = -|\phi\rangle.$$

(a) Show that the charge conjugation operator has eigenvalues of ± 1.

(b) Show that a general eigenstate of the charge operator with charge q,

$$\widehat{\mathcal{Q}}|q\rangle = q|q\rangle,$$

can only be an eigenstate of the charge conjugation operator if $q = 0$. In other words, only neutral particles are eigenstates of the charge conjugation operator!

(c) Assuming that both $|\phi\rangle$ and $|\overline{\phi}\rangle$ are **odd** under parity transformations, find the linear combinations of $|\phi\rangle$ and $|\overline{\phi}\rangle$ that are eigenstates of the combined operation of charge conjugation and parity, $\widehat{\mathcal{C}}\widehat{P}$. Show that their eigenvalues match expectations from part (a) of this question.

(d) Do you expect there to be a conserved quantity associated with symmetry under charge conjugation? Explain your reasoning.

Question 3 **20pts**

Three identical spin-one bosons are in a symmetric spatial state, described by the wavefunction $\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\phi(\mathbf{r}_3)$. Write down all possible normalised spin wavefunctions of the system and the corresponding values of the total spin, projected onto the z-axis.