Coupling to matter

It is natural to couple the gauge potential A_μ to charged sources, j^μ

a functional of matter fields

So starting from our general Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \longrightarrow \mathcal{L}^{(j)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

In this case, Maxwell's equations read

$$\partial_\mu F^{\mu\nu} = j^\nu$$

But $F^{\mu\nu}$ is antisymmetric

$$\Rightarrow \partial_\nu \partial_\mu F^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\nu j^\nu = 0$$

j^ν is a conserved current!

what options do we have?

1. QED

In this case we want to couple A_μ to fermion fields ψ.

Recall that ψ was invariant under

$$\psi(x) \to \psi'(x) = e^{-i\theta} \psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = e^{i\theta} \bar{\psi}(x)$$

and the corresponding current was the vector current

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

Then our Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{\partial} - m)\Psi - g\bar{\Psi}\gamma^\mu \Psi A_\mu$$

which we will write as

$$\boxed{\mathcal{L}^{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi}$$

Here D_μ is the <u>covariant derivative</u>

$$D_\mu \Psi = \partial_\mu \Psi + igA_\mu \Psi.$$

This Lagrangian is invariant under gauge transformations

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x)$$

$$\Psi(x) \to \Psi'(x) = e^{-ig\lambda(x)}\Psi(x)$$

$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = e^{ig\lambda(x)}\bar{\Psi}(x)$$

We can see this because

$$\begin{aligned}
D_\mu \Psi \to D'_\mu \Psi' &= D'_\mu (e^{-ig\lambda(x)} \Psi) \\
&= \partial_\mu (e^{-ig\lambda}\Psi) + ig(A_\mu + \partial_\mu \lambda)(e^{-ig\lambda}\Psi) \\
&= -ig(\partial_\mu \lambda) e^{-ig\lambda}\Psi + e^{-ig\lambda}\partial_\mu \Psi \\
&\quad + e^{-ig\lambda}(igA_\mu \Psi) + ig(\partial_\mu\lambda)e^{-ig\lambda}\Psi \\
&= e^{-ig\lambda}(\partial_\mu \Psi + igA_\mu \Psi) \\
&= e^{-ig\lambda} D_\mu \Psi
\end{aligned}$$

$$\Rightarrow \bar{\Psi}\not{D}\Psi \to \bar{\Psi}'\not{D}'\Psi = \bar{\Psi}e^{ig\lambda}\not{D}\Psi e^{-ig\lambda} = \bar{\Psi}\not{D}\Psi$$

Comments

- Replacing ∂^μ with D^μ in order to couple a $U(1)$ symmetry to gauge fields is called the "<u>principle of minimal coupling</u>"

- We have "promoted" the global symmetry (constant) to a local symmetry $(\lambda(x))$. This is why a gauge symmetry (which is local) seems to have a Noether current (the vector current), which is really associated with the single global symmetry $\lambda(x) = \theta$. This global symmetry is a true symmetry of the theory and not a redundancy in our description.

- The coupling g is the charge of the fermion (and when renormalised, it's the physical charge of the electron)

$$Q = \int d^3\bar{x}\, j^0 = g\int d^3\bar{x}\, \psi^\dagger(x)\,\psi(x)$$

$\uparrow$ N.B. $(\gamma^0)^2 = 1$

$$\rightarrow Q = g\int \frac{d^3\bar{p}}{(2\pi)^3} \sum_{s=1}^{2} \left(a^{s\dagger}(\bar{p})\, a^s(\bar{p}) - b^{s\dagger}(\bar{p})\, b^s(\bar{p}) \right)$$

For QED we usually write

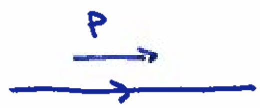
$$\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0} \simeq \frac{1}{137}$$ – (dimensionless) <u>fine structure constant</u>

$e \equiv 1.602\,176\,634 \times 10^{-19}\,C$

$e^2 = g_R^2$

Feynman rules for QED

N.B. $S_F(x-y) = \langle 0 | T\{\Psi(x)\bar{\Psi}(y)\} | 0 \rangle$



$$= \tilde{S}_F(p) = \frac{i(\not{p} - m)}{p^2 - m^2 + i\epsilon}$$

$$= \tilde{G}_F^{\mu\nu}(p) = \frac{-i g_{\mu\nu}}{p^2 + i\epsilon}$$

$$\tilde{\Psi} | \bar{p}, s\rangle = u^s(\bar{p})$$

N.B. Be careful!

$$\Psi(x) | \bar{p}, s\rangle = e^{-ip\cdot x} u^s(\bar{p}) | 0 \rangle$$

but in momentum space momentum conservation is implied (so the $\int d^4x \, e^{-ip\cdot x}$ is already included

$$\langle \bar{p}, s | \tilde{\bar{\Psi}} = \bar{u}^s(p)$$

$$\tilde{\bar{\Psi}} | \bar{p}, s\rangle = \bar{v}^s(\bar{p})$$

$$\langle \bar{p}, s | \tilde{\Psi} = v^s(\bar{p})$$

$$\tilde{A}_\mu | \bar{p}\rangle = \epsilon_\mu(p)$$

In Coulomb gauge $\epsilon^0 = 0$ and $\bar{\epsilon}\cdot\bar{p} = 0$.

$$\langle \bar{p} | \tilde{A}_\mu = \epsilon^*_\mu(p)$$

$$= -ig\gamma^\mu$$

Note: the direction of momentum relative to that of fermion number is important. P+S do not do this, but I recommend always drawing a separate momentum flow arrow for fermion propagators.

If you choose a mode expansion so that $a(\bar{p})e^{-ip\cdot x}$ and $b^{+}(\bar{p})e^{ip\cdot x}$ appear, <u>then</u>:

- internal fermion lines have momentum flowing in the same direction as the fermion arrow
- external ingoing fermions have ingoing momentum
- external outgoing fermions have outgoing momentum
- external ingoing antifermions have outgoing momentum
- external outgoing antifermions have ingoing momentum

A comment on disconnected contributions in scattering amplitudes (one discussed feature of HW5) in David Tong's notes.

David Tong comments (p. 54) that we assume initial and final states in $\langle f|S|i\rangle$ are eigenstates of the free theory. This assumption is tied to the lack of disconnected diagrams in all his results, as he points out on p. 70. In essence, by assuming the initial/final states are free, he assumes the one-particle states are equivalent in the free and interacting theories (hence amputated) and that we can use $|i\rangle \sim |0\rangle$ (hence no disconnected contributions). He discusses this on p. 79. Note that P+S + DT do the same thing, but in different orders. P+S show first that we can drop disconnected diagrams then (implicitly) use this in scattering theory; DT does largely the opposite.

2. Scalar QED

Remember that real scalar fields have no internal symmetri so we cannot couple them to gauge fields. But! Complex scalar fields do have a U(1) symmetry and the associated current is

$$j_\mu = (\partial_\mu \phi^*)\phi - \phi^* \partial_\mu \phi$$

$-ie\, j_\mu A$ is not gauge invariant
the new conserved current will pick up a term $\sim j_\mu A^\mu$ because $j_\mu \sim \partial_\mu \phi \Rightarrow$ inconsisten

Rather than coupling this directly to the gauge field, as we did for (fermionic) QED, we use the principle of minimal coupling to write

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^* D^\mu\phi - m^2|\phi|^2$$

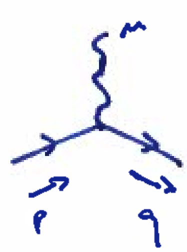
$$D_\mu\phi = \partial_\mu\phi + igA_\mu\phi$$

$$\Rightarrow D_\mu\phi \to e^{-ig\lambda} D_\mu\phi \quad \text{if} \begin{cases} A_\mu \to A_\mu + \partial_\mu\lambda \\ \phi \to e^{-ig\lambda}\phi \end{cases}$$

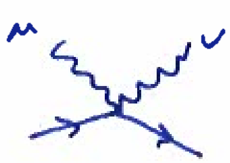
Feynman rules for Scalar QED

$= \tilde{G}_F(p) = \dfrac{i}{p^2 - m^2 + i\epsilon}$

$= \tilde{G}_F^{\mu\nu}(p) = \dfrac{-ig_{\mu\nu}}{p^2 + i\epsilon}$



$= -ig(p+q)_\mu$



$= 2ig^2 g_{\mu\nu}$

$\Big\}$ arise because

$$D_\mu\phi^* D^\mu\phi = -igA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + g^2 A_\mu A^\mu \phi^*\phi$$

Scattering in QED

What is all the fuss about anyway? Why have we done all this work just to get here: calculating scattering amplitudes in QED?

QED occupies a special place in the hearts and history of particle physics – it's really where all of the foundations of 20th century particle physics were laid. QED describes the interactions of charged fermions with photons (ie. EM in QFT language) and is one of the four fundamental forces. It has been spectacularly successful and its success largely drove the efforts to formulate the Standard Model, by applying similar ideas to QCD and the weak force. For example, the anomalous magnetic moment of the electron $a_e = \frac{g-2}{2}$ has been calculated to 10th order in perturbation theory and the theory result is $1\,159\,652\,182.032(720) \times 10^{-12}$. The most precise experimental result is $1\,159\,652\,180.73(28) \times 10^{-12}$

Laporta has spent ~30 years calculating the 8th order mass independent terms to 1100 digits...

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In 1712.06060 they found an error in an integral in the mass-independent 10th order contribution (there are 12672 Feynman diagrams that are mass independent).

~ 9 equivalent diagrams

Physically, QED incorporates three "generations" of leptons

- electrons $m_e \sim 0.51$ MeV → all have spin $1/2$, charge $\pm e$
- muons $m_\mu \sim 106$ MeV
- tauons $m_\tau \sim 1777$ MeV

N.B. neutrinos are leptons, but they are neutral!

To account for these generations, we can modify $\mathcal{L}_{QED}$

$$\mathcal{L}_{QED} = \sum \bar{\psi}_\ell (i\not{\partial} - m_\ell)\psi_\ell - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Here ℓ is a new index (not a spinor index!) representing the type of fermion/lepton - also called its "flavour".

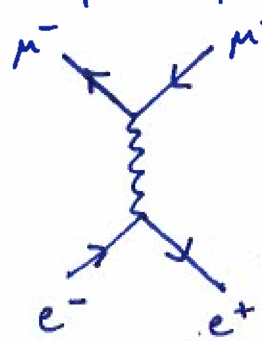
The fact that all leptons listed have the same charge leads to lepton universality - they all interact with photons in the same way. And if the masses were all the same there would be a further symmetry similar to SU(2) isospin in QCD.

The existence of different lepton flavours leads to a multitude of processes - life would be way more boring if all we had were electrons and positrons.

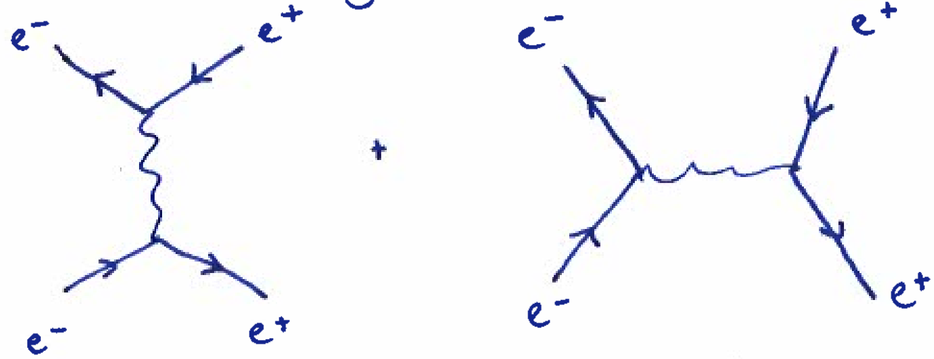
[P+S chap. 5]

Examples

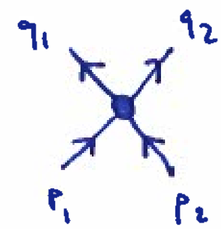
- lepton pair production, e.g. $e^+e^- \to \mu^+\mu^-$



• Bhabha scattering : $e^+e^- \to e^+e^-$

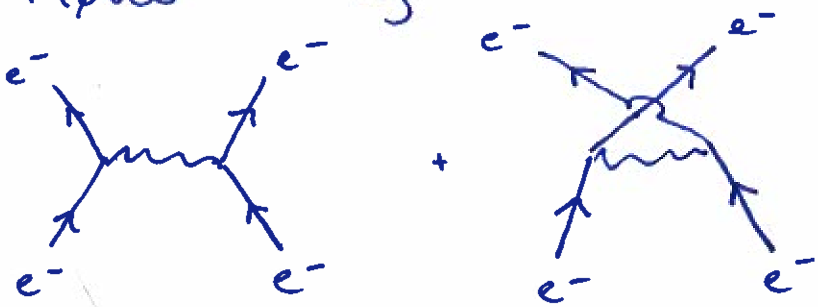


"s-channel" "t-channel" ← "Mandelstam variables"



$s = (p_1 + p_2)^2$
$t = (q_1 - p_1)^2$
$u = (q_1 - p_2)^2$

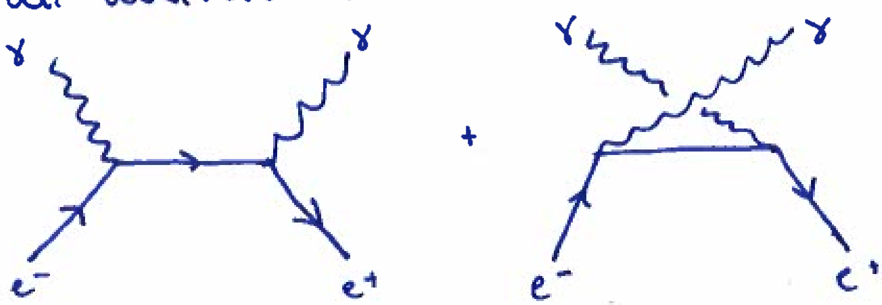
• Møller scattering : $e^-e^- \to e^-e^-$



"u-channel"

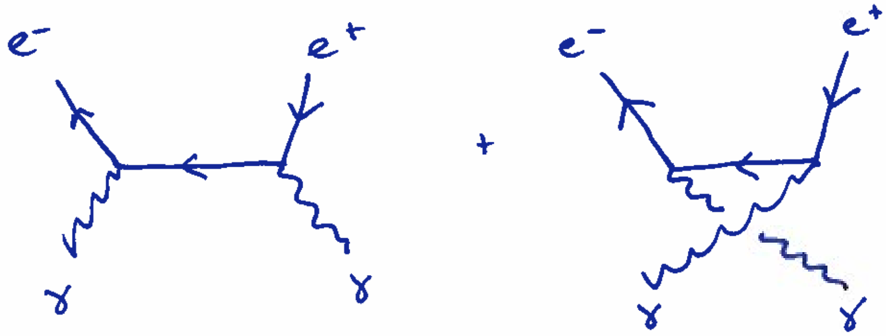
• Compton scattering : $e^-\gamma \to e^-\gamma$

• Pair annihilation : $e^+e^- \to 2\gamma$

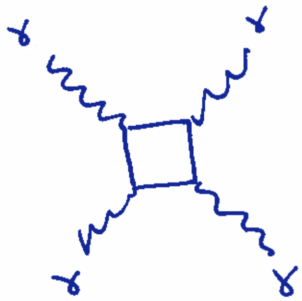


- "Breit-Wheeler process" : $\gamma\gamma' \to e^+e^-$

Predicted 1934
Yet to be observed
in a laboratory!



- Photon scattering : $\gamma\gamma' \to \gamma\gamma'$



← a "box diagram"

<u>Example : unpolarised $e^+e^- \to \mu^+\mu^-$</u>

We will start our study of scattering with the first of these, muon pair production. Specifically, we start by calculating the unpolarised cross-section. $\boxed{\text{P+S } 5.1}$

Recall that the cross-section for two final state particles is

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_A E_B |v_A - v_B|} \frac{|\bar{p}_1|}{(2\pi)^2 4E_{cm}} |\mathcal{M}(p_A, p_B \to p_1, p_2)|^2$$

↑ the <u>invariant matrix element</u>

obtained from the T-matrix as

$$\langle p_1 \cdots p_n | iT | p_A, p_B \rangle = (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum_f p_f\right) i\mathcal{M}(p_A, p_B \to \{p_f\})$$

To get the cross-section our procedure is basically

1. write down the Feynman diagrams

2. use these to calculate the invariant matrix element, $i\mathcal{M}$

3. calculate $|\mathcal{M}|^2$

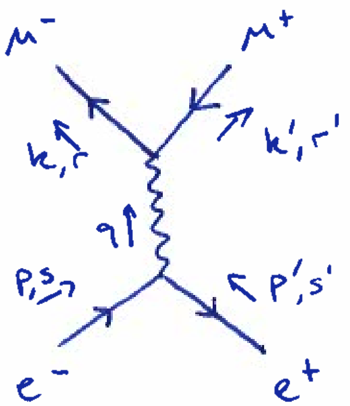
4. use a convenient reference frame to simplify $|\mathcal{M}|^2$

5. calculate the cross-section

So let's just jump right in.

we chose this process partly because there's only one.

$\boxed{1.}$ We know the Feynman diagram



$\boxed{2.}$

$$= \bar{v}(\bar{p}')(-ig\gamma^\mu)u^s(\bar{p})\left(-\frac{ig_{\mu\nu}}{q^2}\right)$$

$$\times\ \bar{u}^r(\bar{k})(-ig\gamma^\nu)v^{r'}(k')$$

$$= \underbrace{ig^2\frac{1}{(p+p')^2}\bar{v}^{s'}(\bar{p}')\gamma^\mu u^s(\bar{p})\cdot\bar{u}^r(\bar{k})\gamma_\mu v^{r'}(\bar{k}')}_{\equiv\ i\mathcal{M}}$$

N.B. $s = (p+p')^2$

$\boxed{3.}$ So how do we calculate $|\mathcal{M}|^2$? Clearly we need $\mathcal{M}^*$

$$(i\mathcal{M})^* = -ig^2\frac{1}{(p+p')^2}\left(v^{s'}(\bar{p}')\gamma^\mu u^s(\bar{p})\right)^\dagger\left(\bar{u}^r(\bar{k})\gamma_\mu v^{r'}(\bar{k}')\right)^*$$

$((\bar{v}\gamma^\mu u)^T)^\dagger$

where

$$(v^{s'}(\bar{p}')\gamma^\mu u^s(\bar{p}))^* = (v^\dagger\gamma^0\gamma^\mu u)^* \overset{\uparrow}{=} u^\dagger\gamma^{\mu\dagger}\gamma^0 v = u^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0 v = \bar{u}\gamma^\mu v$$

$$\Rightarrow (i\mathcal{M})^\dagger = -ig^2\frac{1}{(p+p')^2}\bar{u}^s(\bar{p})\gamma^\mu v^{s'}(\bar{p}')\cdot\bar{v}^{r'}(\bar{k}')\gamma_\mu u^r(\bar{k})$$

so

$$|\mathcal{M}|^2 = \frac{g^4}{(p+p')^4}\bar{u}^s(\bar{p})\gamma^\mu v^{s'}(p')\cdot\bar{v}^{s'}(\bar{p}')\gamma^\nu u_s(\bar{p})\cdot\bar{v}^{r'}(\bar{k}')\gamma_\mu u^r(\bar{k})\cdot\bar{u}^r(\bar{k})\gamma_\nu v^{r'}(\bar{k}')$$

The simplest quantity we can calculate is the <u>unpolarised cross-section</u> – extracted from unpolarised beams in an experiment where we don't measure the polarisation of the final state fermions – which means we:

- <u>average</u> over <u>initial state spins</u>
- <u>sum</u> over <u>final state spins</u>

$$\Rightarrow \overline{|M|^2} = \left(\frac{1}{2}\sum_s\right)\left(\frac{1}{2}\sum_{s'}\right)\sum_r\sum_{r'}|M(ss'\to rr')|^2$$

$$= \frac{1}{4}\frac{g^4}{(p+p')^4}\sum_{ss'}\bar{u}^s(\bar{p})\gamma^\mu v^{s'}(\bar{p}')\,\bar{v}^{s'}(\bar{p})\gamma^\nu u^s(\bar{p})$$

$$\times \sum_{rr'}\bar{v}^{r'}(\bar{k}')\gamma_\mu u^r(\bar{k})\,\bar{u}^r(\bar{k})\gamma_\nu v^{r'}(\bar{k}')$$

Recall our outer product identities! They finally come in useful

$$\sum_{s'} v^{s'}_\beta(\bar{p}')\bar{v}^{s'}_\gamma(\bar{p}') = (\not{p}'-m_e)_{\beta\gamma} \qquad \sum_r u^r_\beta(\bar{k})\bar{u}^r_\gamma(\bar{k}) = (\not{k}+m_\mu)_{\beta\gamma}$$

$$\sum_s u^s_\delta(\bar{p})\bar{u}^s_\alpha(\bar{p}) = (\not{p}+m_e)_{\delta\alpha} \qquad \sum_{r'}\bar{v}^{r'}_\delta(\bar{k}')\bar{v}^{r'}_\alpha(\bar{k}') = (\not{k}'-m_\mu)_{\delta\alpha}$$

$$\Rightarrow \overline{|M|^2} = \frac{1}{4}\frac{g^4}{(p+p')^4}\left[(\not{p}+m_e)^{\delta\alpha}\gamma^\mu_{\alpha\beta}(\not{p}'-m_e)^{\beta\gamma}\gamma^\nu_{\gamma\delta}\right]$$

$$\times\left[(\not{k}'-m_\mu)_{\delta'\alpha'}\gamma_\mu^{\alpha'\beta'}(\not{k}+m)_{\beta'\gamma'}\gamma_\nu^{\gamma'\delta'}\right]$$

$$= \frac{1}{4}\frac{g^4}{(p+p')^4}\mathrm{Tr}\left[(\not{p}+m_e)\gamma^\mu(\not{p}'-m_e)\gamma^\nu\right]$$

$$\times\mathrm{Tr}\left[(\not{k}'-m_\mu)\gamma_\mu(\not{k}+m)\gamma_\nu\right]$$

So now all we need to do is calculate a bunch of traces...

Thankfully there are a few tricks we can use.

→ the "<u>trace identities</u>"

These identities follow from the properties of γ matrices

↑ one reason I summarised them in one place earlier

- $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
- $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
- $\{\gamma^\mu, \gamma^5\} = 0$
- $(\gamma^5)^2 = \mathbb{1}$

Identities

1. $\mathrm{Tr}(\gamma^\mu) = 0$
2. $\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) = 0$ if n odd
3. $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
4. $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$
5. $\mathrm{Tr}(\gamma^5) = 0$
6. $\mathrm{Tr}(\gamma^5\gamma^{\mu_1}\ldots\gamma^{\mu_n}) = 0$ for $n \leqslant 3$
7. $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}$

Let's prove a couple of others and leave the rest for HW.

1. $\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu) \overset{\text{cyclic}}{=} \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^5)$

$= -\mathrm{Tr}(\gamma^\mu\gamma^5\gamma^5) = -\mathrm{Tr}(\gamma^\mu) \Rightarrow = 0$

↑ similar argument works for n odd

3. $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)$
$= 2g^{\mu\nu}\,\mathrm{Tr}\,\mathbb{1} - \mathrm{Tr}(\gamma^\nu\gamma^\mu)$
$= 8g^{\mu\nu} - \mathrm{Tr}(\gamma^\mu\gamma^\nu)$

$\Rightarrow 2\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 8g^{\mu\nu}$
$\Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

repeated application of similar ideas works for $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$

Now let's use these tricks in our invariant matrix element

$$\mathrm{Tr}\left[(\not p + m_e)\gamma^\mu(\not p' - m_e)\gamma^\nu\right] = \mathrm{Tr}\left[\not p\gamma^\mu\not p'\gamma^\nu + m_e\gamma^\mu\not p'\gamma^\nu + \not p\gamma^\mu(-m_e)\gamma^\nu + m_e\gamma^\mu(-m_e)\gamma^\nu\right]$$

$$= p_\alpha p'_\beta \mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] + m_e p'_\alpha \mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\nu\right] - m_e p_\alpha \mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\nu\right] - m_e^2\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]$$

$$= p_\alpha p'_\beta\, 4\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta}\right) - m_e^2\, 4g^{\mu\nu}$$

$$= 4\left(p^\mu p'^\nu - p\cdot p' g^{\mu\nu} + p^\nu p'^\mu - m_e^2 g^{\mu\nu}\right)$$

$$= 4\left(p^\mu p'^\nu + p^\nu p'^\mu - (p\cdot p' + m_e^2)g^{\mu\nu}\right)$$

Similarly we get

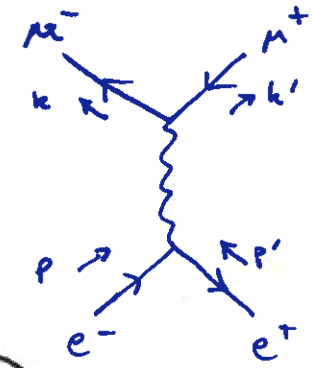
$$\mathrm{Tr}\left[(\not k' - m_{(\mu)})\gamma_\mu(\not k + m_{(\mu)})\gamma_\nu\right] = 4\left(k_\mu k'_\nu + k_\nu k'_\mu - (k\cdot k' + m_{(\mu)}^2)g_{\mu\nu}\right)$$

Thus our invariant matrix element is

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\frac{g^4}{(p+p')^4}\, 4\left(p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)\right) \times 4\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_{(\mu)}^2)\right)$$

Reminder

$e^+e^- \to \mu^+\mu^-$



1. Feynman rules
2. Invariant matrix element, iM
3. $|M|^2$
4. Choose reference frame
5. Calculate observable

$$iM = \frac{ig^2}{(p+p')^2}\,\bar{v}^{s'}(\bar{p}')\gamma^\mu u^s(\bar{p})\cdot\bar{u}^r(\bar{k})\gamma_\mu v^{r'}(\bar{k}')$$

$$\overline{|M|^2} = \frac{8g^4}{(p+p')^4}\Big[p\cdot k\; p'\cdot k' + p\cdot k'\,p'\cdot k + m_e^2\, k\cdot k' + m_\mu^2\, p\cdot p' + 2m_e^2 m_\mu^2\Big]$$

So now we need to move onto $\boxed{4.}$

$$\overline{|M|^2} = \frac{4g^4}{(p+p')^4}\Big[\, p\cdot k\, p'\cdot k' + p\cdot k'\, p'\cdot k - p\cdot p'\left(k\cdot k' + m_{(\mu)}^2\right)$$
$$+ p'\cdot k\, p\cdot k' + p\cdot k\, p'\cdot k' - p\cdot p'\left(k\cdot k' + m_{(\mu)}^2\right)$$
$$- 2k\cdot k'\left(p\cdot p' + m_e^2\right) + 4\left(p\cdot p' + m_e^2\right)\left(k\cdot k' + m_{(\mu)}^2\right)\Big]$$

$\uparrow\ g^{\mu\nu}g_{\mu\nu} = g^{\mu}{}_{\mu} = 4$

$$= \frac{4g^4}{(p+p')^4}\Big[\,2p\cdot k\, p'\cdot k' + 2p\cdot k'\, p'\cdot k - 2p\cdot p'\left(k\cdot k' + m_{(\mu)}^2\right)$$
$$- 2k\cdot k'\left(p\cdot p' + m_e^2\right) + 4\left(p\cdot p' + m_e^2\right)\left(k\cdot k' + m_{(\mu)}^2\right)\Big]$$

$$= \frac{4g^4}{(p+p')^4}\Big[\,2p\cdot k\, p'\cdot k' + 2p\cdot k'\, p'\cdot k + 2m_{(\mu)}^2\, p\cdot p' + 2m_e^2\, k\cdot k' + 4m_e^2 m_{(\mu)}^2\Big]$$

$$= \frac{8g^4}{(p+p')^4}\Big[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_e^2\, k\cdot k' + m_{(\mu)}^2\, p\cdot p' + 2m_e^2 m_{(\mu)}^2\Big]$$

↑ N.B. it's Lorentz invariant!

$\boxed{4.}$ Now we choose a convenient reference frame, in particular we will choose the <u>centre of momentum frame</u> (CM) $\Rightarrow \boxed{\sum \bar{p} = 0}$

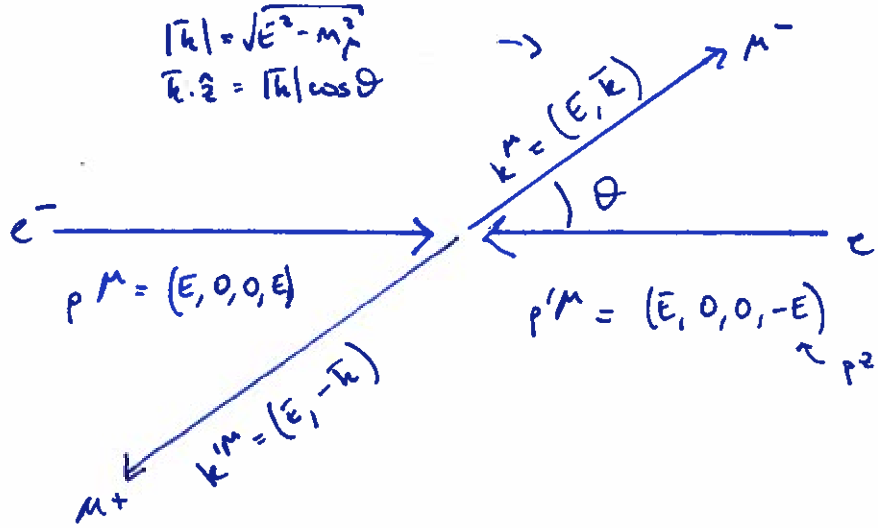
We will also simplify our lives by assuming $\frac{m_e}{E} \ll 1$.

↑ recall $m_e \sim 0.5$ MeV!

Then our collision looks like

$|\bar{k}| = \sqrt{E^2 - m_\mu^2}$

$\bar{k}\cdot\hat{z} = |\bar{k}|\cos\theta$



↖ $p^z = E$ because $m_e \approx 0$

Q: What energy do we need for this error to be $O(g^6)$? ~ 70 MeV

Now we can simplify the scalar products

$$(p+p')^2 = (2E)^2 = 4E^2$$
$$p\cdot p' = E^2 - E(-E) = 2E^2$$
$$p\cdot k = E^2 - E|\bar{k}|\cos\vartheta$$
$$p'\cdot k' = E^2 - E|\bar{k}|\cos\vartheta$$
$$p\cdot k' = E^2 + E|\bar{k}|\cos\vartheta$$
$$p'\cdot k = E^2 + E|\bar{k}|\cos\vartheta$$

We plug these into our invariant matrix element

$$\overline{|M|^2} = \frac{8g^4}{(4E^2)^2}\left[(E^2 - E|\bar{k}|\cos\vartheta)^2 + (E^2 + E|\bar{k}|^2\cos\vartheta)^2 + m_{(\mu)}^2\cdot 2E^2\right]$$
$$= \frac{g^4}{2E^4}E^2\left[(E - |\bar{k}|\cos\vartheta)^2 + (E + |\bar{k}|\cos\vartheta)^2 + 2m_{(\mu)}^2\right]$$
$$= \frac{g^4}{2E^2}\left[2E^2 + 2|\bar{k}|^2\cos^2\vartheta + 2m_{(\mu)}^2\right]$$
$$= g^4\left[1 + \frac{m_{(\mu)}^2}{E^2} + \frac{|\bar{k}|^2\cos^2\vartheta}{E^2}\right] \qquad |\bar{k}|^2 = E^2 - m_{(\mu)}^2$$
$$= g^4\left[1 + \frac{m_{(\mu)}^2}{E^2} + \left(1 - \frac{m_{(\mu)}^2}{E^2}\right)\cos^2\vartheta\right]$$

$\boxed{5.}$ Finally we can substitute our expression for $\overline{|M|^2}$ into

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{1}{4E_pE_{p'}}\frac{1}{|v_p - v_{p'}|}\frac{1}{16\pi^2}\frac{|\bar{k}|}{E_{cm}}\overline{|M|^2}$$

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In this case $v_p - v_{p'} = v_{e^-} - v_{e^+}$
$$= \frac{p^z}{E} - \frac{p'^z}{E'}$$
$$= \frac{E}{E} - \frac{(-E)}{E} = 2$$

So we have

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{1}{4E^2}\frac{1}{2}\frac{1}{16\pi^2}\frac{\sqrt{E - M_{(\mu)}^2}}{2E}\, g^4\left[1 + \frac{M_{(\mu)}^2}{E^2} + \left(1 - \frac{M_{(\mu)}^2}{E^2}\right)\cos^2\theta\right]$$

$$= \frac{\alpha^2}{16E^2}\sqrt{1 - \frac{M_{(\mu)}^2}{E^2}}\left[1 + \frac{M_{(\mu)}^2}{E^2} + \left(1 - \frac{M_{(\mu)}^2}{E^2}\right)\cos^2\theta\right]$$

We can consider two limits

- high-energy $\frac{M_{(\mu)}^2}{E} << 1$ $\qquad \left.\frac{d\sigma}{d\Omega}\right|_{cm} \simeq \frac{\alpha^2}{16E^2}\left(1 + \cos^2\theta\right)$

- threshold $\frac{M_{(\mu)}^2}{E^2} \simeq 1$ $\qquad \left.\frac{d\sigma}{d\Omega}\right|_{cm} \simeq \frac{\alpha^2}{16M_{(\mu)}^2}\frac{|\vec{k}|}{M_{(\mu)}^2}$

Q: What happens for $\frac{M_{(\mu)}^2}{E^2} >> 1$?

See P+S. p. 137 – 140 for more information on these results and for an introduction to $e^+e^- \to q\bar{q}$ – an extremely important experimental process