Thursday 15

Recall:

Lorentz transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Scalar field $\phi(x) \to \phi'(x') = \phi(x)$

Vector field $A^\mu(x) \to A^{\mu\prime}(x') = \Lambda^\mu{}_\nu A^\nu(x)$

Weyl spinor $\psi_{L/R}(x) \to \psi'_{L/R}(x') = \Lambda_{L/R}\psi_{L/R}(x)$

Here $\Lambda_{L/R} = \exp\{(-i\bar\theta \mp \bar\eta)\cdot\bar\sigma\}$ $\quad$ $\Lambda_{L/R} = \exp(\frac{1}{2}(i\bar\theta \mp \bar\beta)\cdot\bar\sigma)$ in Schwartz 10.38, 10.39

$\theta^i = \frac{1}{2}\epsilon^{ijk}\omega^{jk}$, $\eta^i = \omega^{i0}$

Are we sure vectors and spinors are different? Tong works out an example p. 85, see 4.1.1. Schwartz also, in Sec 10.5.

but note $\psi^\alpha(x) \to S(\Lambda)^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x)$ ← $S(\Lambda)$ in Tong 4.22

Dirac spinor $\psi(x) \to \psi'(x') = \Lambda_D \psi(x)$

$S^{\mu\nu}$ in P+S 3.2, $S^{\mu\nu} = \frac{\sigma^{\mu\nu}}{2}$

← $S^{\rho\sigma}$ in Tong 4.18 up to i

Here $\Lambda_D = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} = \exp\{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\}$

in chiral basis $\quad$ "$J^{\mu\nu} = \frac{\sigma^{\mu\nu}}{2}$" $\quad$ $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$S^{\mu\nu}$ in Schwartz 10.68

Let's visit Lorentz transformations and the Lorentz group

We can write infinitesimal transformations as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \quad \text{with} \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$ in Tong 4.10

3 rotation angles
3 velocity components (boosts)

Lorentz group is $O(3,1)$ and proper orthochronous Lorentz transformations are $SO(3,1)$ with $\Lambda^0{}_0 \geq 1$.

$\det\Lambda = +1$

Generators of the Lorentz Lie algebra are $J^{\mu\nu}$

$$\Rightarrow \quad \Lambda = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}$$

$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma} M^{\rho\sigma}\right) \leftarrow$ Tong 4.1

An object ϕ^i transforms under a representation of the Lorentz group if it obeys

generator in representation R

$\nwarrow (J_R^{\mu\nu})^i{}_j$ an $n \times n$ matrix

$$\phi^i \longrightarrow \underbrace{\left[\exp\left(-\tfrac{i}{2}\omega_{\mu\nu} J_R^{\mu\nu}\right)\right]^i{}_j}_{\text{matrix representation of dimension } n} \phi^j$$

Some representations

$(\gamma^{\mu\nu})_{\alpha\beta}$ in P+S eq 3.18

(0,0) $\rightarrow$ $n = 1$, scalar : $J^{\mu\nu} = 0$

$(\frac{1}{2},\frac{1}{2}) \rightarrow$ $n = 4$, vector : $(J^{\mu\nu})^\rho{}_\sigma = i\left(g^{\mu\rho}\delta^\nu{}_\sigma - g^{\nu\rho}\delta^\mu{}_\sigma\right)$

$\nwarrow (M^{\rho\sigma})^\mu{}_\nu$ in Tong eq 4.8 up to factor i

...posing ...ality sing ...$\frac{1}{2},\frac{1}{2}$) $\mathbb{C}$ ...-vector rep ... $\mathbb{R}$ 4-vector rep

$\swarrow$ satisfies

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)$$

$\uparrow$ Lorentz Lie algebra

We can write these generators as

P+S : 3.26 and 3.27

$S^{0i} = -\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}$

$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}$,

c.f. Tong 4.25 and 4.27

$$J^i = \tfrac{1}{2}\epsilon^{ijk}J^{jk} \qquad K^i = J^{i0}$$

$\Rightarrow [J^i, J^j] = i\epsilon^{ijk}J^k \quad \leftarrow SU(2)$! J^i is an angular momentum

$[J^i, K^j] = i\epsilon^{ijk}K^k \quad \leftarrow K^i$ is a spatial vector

$[K^i, K^j] = -i\epsilon^{ijk}J^k$

Schwartz 10.17, 10.18, 10.19

Now we define

$$\vartheta^i = \frac{1}{2}\epsilon^{ijk}\omega^{k} \qquad \eta^i = \omega^{i0}$$

$$\Rightarrow \frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} = \bar{\vartheta}\cdot\bar{J} - \bar{\eta}\cdot\bar{K}$$

$$\Rightarrow \Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} = e^{-i\bar{\vartheta}\cdot\bar{J} + i\bar{\eta}\cdot\bar{K}}$$

Question: show that this definition leads to the correct (anticlockwise) rotations and boosts

$$\delta x^\mu = x'^\mu - x^\mu = \Lambda^\mu{}_\nu x^\nu - x^\mu = (-i\vartheta^3 J^3)^\mu{}_\nu x^\nu$$

$$= -i\underbrace{\left(\tfrac{1}{2}\epsilon^{3jk}\omega^{jk}\right)}_{\frac{1}{2}\epsilon^{312}\omega^{12}+\frac{1}{2}\epsilon^{321}\omega^{21} = \epsilon^{312}\omega^{12} = \vartheta^3}\left(\tfrac{1}{2}\epsilon^{3jk}J^{jk}\right)^\mu{}_\nu x^\nu$$

and similar

$$= -i\vartheta^3 (J^{12})^\mu{}_\nu x^\nu = -i\vartheta\left(ig^{1\mu}\delta^2{}_\nu - ig^{2\mu}\delta^1{}_\nu\right)x^\nu$$

$\Rightarrow \delta x = -\vartheta\,\delta y \qquad \delta y = -\vartheta\,\delta x$ anticlockwise rotation

$$\delta x^\mu = (i\eta^1 K^1)^\mu{}_\nu x^\nu$$

$$= i(\omega^{10}J^{10})^\mu{}_\nu x^\nu$$

$$= i\eta\left(ig^{1\mu}\delta^0{}_\nu - ig^{0\mu}\delta^1{}_\nu\right)x^\nu$$

$\Rightarrow \delta t = -\eta\,\delta x \qquad \delta x = +\eta\,\delta t$ boost in x-direction

□

$n=2$, spinor : To study spinor representations, we introduce generators

$(\frac{1}{2},0)$ and $(0,\frac{1}{2})$

Schwartz 10.23

$$J^{i\pm} = \frac{J^i \pm i K^i}{2}$$

which satisfy

$$[J^{+i}, J^{+j}] = i\epsilon^{ijk} J^{+k}$$
$$[J^{-i}, J^{-j}] = i\epsilon^{ijk} J^{-k}$$
$$[J^{+i}, J^{-j}] = 0$$

So we have <u>two</u> independent copies of $SU(2)$ generators

$$\Rightarrow \quad sl(2,\mathbb{C}) \times sl(2,\mathbb{C}) \simeq so(3,1)$$

Thus, representations of the Lorentz Lie algebra can be written as pairs (j_-, j_+), with dimension $(2j_- + 1)(2j_+ + 1)$. Since $J^i = J^{-i} + J^{+i}$, we have

generator of rotations

states in (j_-, j_+) labelled by j in integer steps between $|j_+ - j_-|$ and $j_+ + j_-$.

spin

Representation $(\frac{1}{2}, 0)$ are left-handed two-spinors
– note $j = \frac{1}{2}$ and dimension $= (2\cdot\frac{1}{2}+1)\cdot 1 = 2$

Obvious solution to commutation relations is

$$J^{-i} = \frac{\sigma^i}{2} \qquad J^{+i} = 0 \qquad \Rightarrow J^i = \frac{\sigma^i}{2} \qquad K^i = i\frac{\sigma^i}{2}$$

<u>This</u> is how we obtain $\psi_L \to \Lambda_L \psi_L$

for $(0,\frac{1}{2})$, switch J^- and J^+ etc.

(d)

$n = 4$, spinor :
$\uparrow$
$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

So far we've only consider continuous Lorentz transformations - we now need to move out of the proper orthochronous Lorentz group into the other "pieces". We will consider the role of parity.

Note that :

$P: K^i \to -K^i$ and $P: J^i \to J^i$

$\uparrow$ "true vector" $\qquad$ $\uparrow$ pseudovector

$\Rightarrow P: J^{\pm i} \to J^{\mp i} \Rightarrow P: (j_-, j_+) \to (j_+, j_-)$

$\Rightarrow \psi_{L/R}$ are not eigenstates of the parity operator!

$\uparrow$ better to think of $\psi_{L/R}$ as separately not a basis

But

Allows us to construct objects invariant under parity - required for QED

$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ is a basis for parity

But not weak force!

$$P: \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \to \begin{pmatrix} \psi_R(x') \\ \psi_L(x') \end{pmatrix} \quad \text{with} \quad x'^\mu = (t, -\bar{x})$$

$\uparrow$ or $\psi(x) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi(x')$

Note $C: \psi(x) \to \psi^c(x) = \begin{pmatrix} -i\sigma^2 \psi_R^* \\ i\sigma^2 \psi_L^* \end{pmatrix} = -i \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \psi^*(x)$

$\uparrow$ satisfies $(\psi^c)^c = \psi$

Majorana spinors satisfy $\psi_M^c = \psi_M$, $\psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2 \psi_L^* \end{pmatrix}$

particularly useful for ultra-relativistic/massless case

In the chiral basis, the Dirac equation becomes

$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\qquad$ $(i\not{\partial} - m)\psi = 0$

$$\begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$$\Rightarrow \begin{array}{l} \overbrace{i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})}^{i\sigma^\mu\partial_\mu}\psi_R = m\psi_L \\ \underbrace{i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})}_{i\bar{\sigma}^\mu\partial_\mu}\psi_L = m\psi_R \end{array} \Bigg\}$$

if $m=0$ these decouple! A massless Dirac spinor is two Weyl spinors (uncoupled)

Recall $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ $\left[= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ in chiral rep.}\right]$

so $\left.\begin{array}{l} P_L\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \\ P_R\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \end{array}\right\}$ $P_{L/R} = \dfrac{1 \mp \gamma^5}{2}$ are chiral projectors

← PTO →

Discuss helicity here

We can define new Dirac spinors

$\psi'(x) = U\psi(x)$ $\quad$ for U a constant unitary matrix

preserves $\mathcal{L}$ since

$\bar{\psi}' = \psi'^\dagger\gamma^{0\prime}$

$$\mathcal{L} = \psi'^\dagger U\gamma^0(i\gamma^\mu\partial_\mu - m)U^\dagger\psi' = \bar{\psi}'(i\gamma^{\mu\prime}\partial_\mu - m)\psi' = \mathcal{L}'$$

$\gamma^{\mu\prime} = U\gamma^\mu U^\dagger$

Clifford algebra invariant under this transformation

Weyl spinors are helicity eigenstates when $m = 0$

$$\bar{\sigma}^\mu \partial_\mu \psi_L = 0$$
$$\sigma^\mu \partial_\mu \psi_R = 0$$

Consider ψ_L and let $\psi_L(x) = u_L e^{-ip\cdot x}$

$$\Rightarrow \quad (-iE - i\bar{p}\cdot\vec{\sigma})u_L = 0$$

Since $E^2 = |\bar{p}|^2 \Rightarrow E = |\bar{p}|$ so

$$\frac{\bar{p}}{|\bar{p}|}\cdot\vec{\sigma}\, u_L = -u_L \quad , \quad \bar{J} = \frac{\bar{\sigma}}{2} \text{ and } \hat{p} = \frac{\bar{p}}{|\bar{p}|}$$

$$\Rightarrow \underbrace{(\hat{p}\cdot\bar{J})}_{h} u_L = -\frac{1}{2}u_L$$

$\Rightarrow$ left-handed massless Weyl spinor is a helicity eigenstate with $h = -1/2$

Similarly $h\, u_R = +\frac{1}{2} u_R$

Helicity
- tells us whether spin is (anti)aligned with the particle's momentum
- frame dependent except for massless particles

Chirality
- intrinsic property of fields that tells us how they transform under Lorentz transformations
- massless fermions : chiral subspaces are eigenspaces of helicity

useful in nonrelativistic limit

If we choose

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$ we obtain the Dirac basis

$$\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_L + \psi_R \\ \psi_R - \psi_L \end{pmatrix} \equiv \begin{pmatrix} \phi \\ \chi \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Dirac Lagrangian is

$$\bar{\Psi}(i\not{\partial} - m)\Psi = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$

manifests chiral invariance for $m = 0$

Chiral symmetry: $$\left.\begin{aligned} \psi_L &\to \psi_L' = e^{i\theta_L}\psi_L \\ \psi_R &\to \psi_R' = e^{i\theta_R}\psi_R \end{aligned}\right\} U(1)\times U(1)$$

If $\theta_L = \theta_R = \alpha$ then $\psi \to \psi' = e^{i\alpha}\psi$ ← global U(1) vector

If $\theta_L = -\theta_R = -\beta$ then $\psi \to \psi' = e^{i\beta\gamma^5}\psi$ ← global U(1) axial (or "chiral")

Mass term breaks $U(1)_A$ but not $U(1)_V$.

Solutions of the Dirac equation

We'll assume solutions of the form

$$\psi(x) = u(p)\, e^{-ip\cdot x}$$
$$\psi(x) = v(p)\, e^{+ip\cdot x}$$

$$\Rightarrow (\not{p} - m)\, u(p) = 0$$
$$(\not{p} + m)\, v(p) = 0$$

In the chiral representation, we can write

$$u(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix}$$

Taking $m \neq 0$, $\bar{p} = 0$ we have

$$m(\gamma^0 - 1)\, u(p) = 0 \quad \Rightarrow \quad u_L = u_R$$

$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\hookrightarrow$ choose $u_L = u_R = \sqrt{m}\, \xi$

$\xi^\dagger \xi = 1$

Now boost in p^z

$$u(p) = \exp\left(-\frac{\eta}{2}\cdot \vec{\sigma}\right) u(p) = \exp\left(-\frac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right) u(p)$$

algebra in P+S 3.49

$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i]$
$= -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

3.26 of P+S

and

$\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}}$

3.30 of P+S

$$= \begin{pmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\xi \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\xi \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi \\ \sqrt{p\cdot\bar{\sigma}}\, \xi \end{pmatrix}$$

Now we choose a basis ξ^s

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \sigma^3 \xi^1 = \xi^1 \qquad \sigma^3 \xi^2 = -\xi^2$$

$$\left(\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right.$$

Then terms like

$$\left(\frac{1 \pm \sigma^3}{2} \right) \xi^1 = \xi^1 \text{ or } 0$$

$$\Rightarrow u(p) = \begin{pmatrix} \sqrt{E - p^3}\, \xi^1 \\ \sqrt{E + p^3}\, \xi^1 \end{pmatrix} \xrightarrow{E \sim p^3} \begin{pmatrix} 0 \\ \sqrt{2E}\, \xi^1 \end{pmatrix}$$

"large boost" $\Rightarrow m \ll p^3 \Rightarrow E \sim p^3$

$$\left(\frac{1 \mp \sigma^3}{2} \right) \xi^2 = \xi^2 \text{ or } 0$$

$$\Rightarrow u(p) = \begin{pmatrix} \sqrt{E + p^3}\, \xi^2 \\ \sqrt{E - p^3}\, \xi^2 \end{pmatrix} \xrightarrow{E \sim p^3} \begin{pmatrix} \sqrt{2E}\, \xi^2 \\ 0 \end{pmatrix}$$

For $v(p)$ we ultimately find $\leftarrow$ useful HW Q!

$$v(p) = \begin{pmatrix} \left[\sqrt{E + p^3} \left(\frac{1 - \sigma^3}{2} \right) + \sqrt{E - p^3} \left(\frac{1 + \sigma^3}{2} \right) \right] \eta \\ -\left[\sqrt{E + p^3} \left(\frac{1 + \sigma^3}{2} \right) + \sqrt{E - p^3} \left(\frac{1 - \sigma^3}{2} \right) \right] \eta \end{pmatrix}$$