The Dirac field

So far our entire discussion has focussed on scalar fields. But most of our Universe is not composed of scalar fields, as far as we know. Many fundamental particles are fermions, ie. spin ½ particles. We will arrive at a description of fermions via a seemingly unrelated approach: by studying the Lorentz group. It turns out that the symmetries of spacetime are intimately tied to the spin of various fields.

The Higgs boson is the only known fundamental (?!) scalar particle.

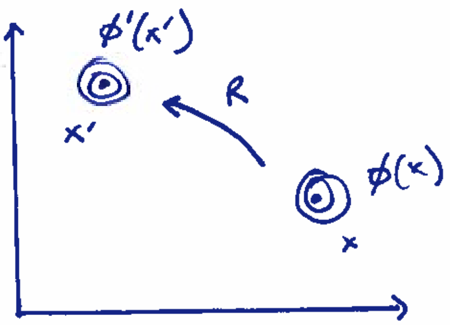
Let's start with the scalar field we know and love. By definition – it's what scalar means – a scalar field is invariant under a Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, so

$$\phi(x) \longrightarrow \phi'(x' = \Lambda x) = \phi(x)$$

transformed field at transformed point equals original field at original point.

Example

Consider a rotation $x' = Rx$



Note: P+S discuss $\phi'(x)$, the transformed field at the original point, $\phi'(x) = \phi(\Lambda^{-1}x)$

What about our scalar field Lagrangian?

First, recall Lorentz transformation properties

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$
$$x'_{\mu} = x_{\nu} (\Lambda^{-1})^{\nu}{}_{\mu}$$

$$\begin{cases} x'_{\mu} x'^{\mu} = x_{\mu} x^{\mu} \\ \quad = x_{\nu} M^{\nu}{}_{\mu} \Lambda^{\mu}{}_{\rho} x^{\rho} \\ \quad = x_{\nu} \delta^{\nu}{}_{\rho} x^{\rho} \\ \Rightarrow M^{\nu}{}_{\mu} \Lambda^{\mu}{}_{\rho} = \delta^{\nu}{}_{\rho} \\ \quad \Rightarrow M^{\nu}{}_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu} \end{cases}$$

So $\partial'_{\mu} \phi'(x') = (\Lambda^{-1})^{\nu}{}_{\mu} \partial_{\nu} \phi(x)$

$\partial'^{\mu} \phi'(x') = \Lambda^{\mu}{}_{\nu} \partial^{\nu} \phi(x)$

Thus

$$\mathcal{L}'(x') = \frac{1}{2}\left(\partial'_{\mu}\phi'(x')\partial'^{\mu}\phi'(x') - m^2 \phi'(x')\phi'(x')\right)$$
$$= \frac{1}{2}\left(\underbrace{(\Lambda^{-1})^{\nu}{}_{\mu} \Lambda^{\mu}{}_{\rho}}_{\delta^{\nu}{}_{\rho}} \partial_{\nu}\phi(x)\partial^{\rho}\phi(x) - m^2\phi(x)\phi(x)\right)$$
$$= \frac{1}{2}\left(\partial_{\nu}\phi(x)\partial^{\nu}\phi(x) - m^2\phi(x)\phi(x)\right) = \mathcal{L}(x)$$

$\Rightarrow$ the Lagrangian (density) is invariant, ie. a Lorentz scalar.

N.B. Not just a trivial property of dot products
$\mathcal{L}(x) = c_{\mu}\partial^{\mu}\phi(x)$
$\rightarrow \Lambda^{\mu}{}_{\nu} c_{\mu} \partial^{\nu}\phi \neq \mathcal{L}$

This is all fine for scalar fields, but what about more complicated objects?

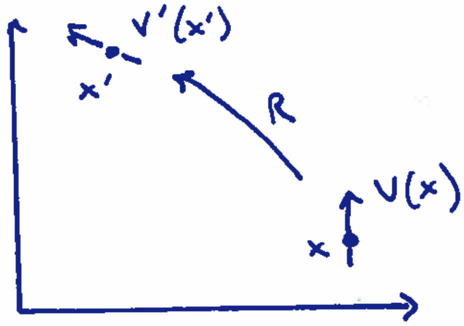
consider a vector field

$$V_i(x) \longrightarrow V'_i(x') = R_{ij} V_j(x) \quad (\neq V_i(x))$$

transformed by a rotation!

Example

Consider a rotation $x' = Rx$



For a general Lorentz transformation

$$V^\mu(x) \to V'^\mu(x'=\Lambda x) = \Lambda^\mu{}_\nu V^\nu(x) \qquad (\neq V(x))$$

And for arbitrary indices (tensors)

$$T^{\mu_1 \cdots \mu_n}(x) \to T'^{\mu_1 \cdots \mu_n}(x'=\Lambda x) = \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1 \cdots \nu_n}(x)$$

Thus we say scalars, vectors, and tensors are all <u>representations of the Lorentz group</u>

These representations (or "reps") are restricted by the properties of the Lorentz group. Let's consider an n-component field $\Phi_{\alpha=1\ldots n}$

not a Lorentz index

This will transform under a Lorentz transformation as

$$\Phi_\alpha(x) \to \Phi'_\alpha(x'=\Lambda x) = S_{\alpha\beta}(\Lambda)\Phi_\beta(x) \qquad (\textstyle\sum_\beta \text{ implied})$$

or just

$$\Phi'_\alpha(x') = S(\Lambda)\Phi(x)$$

If we apply successive transformations, these should be the same as a single composite transformation

group multiplication rule

$$\Phi'(x') = S(\Lambda_2)S(\Lambda_1)\Phi(x) = S(\Lambda)\Phi(x) \Leftrightarrow \begin{cases} S(\Lambda) = S(\Lambda_2)S(\Lambda_1) \\ x' = \Lambda_2\Lambda_1 x \end{cases}$$

In fact, a general representation of the Lorentz group, ie. of $S(\Lambda)$, can be written in terms of the generators $J^{\mu\nu}$ and some parameters $\omega_{\mu\nu}$ as

$$\boxed{\begin{aligned} S(\Lambda(\omega)) &= e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \\ [J^{\mu\nu}, J^{\rho\sigma}] &= i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)\end{aligned}}$$

- $J^{\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$ } because of
- Six independent generators corresponding to three rotations and three boosts
- Commutation relation defines the Lie algebra of the group

  ↑ required to ensure $S(\Lambda) = S(\Lambda_2)S(\Lambda_1)$

One possible representation is the vector representation

HOMEWORK: CHECK THIS SATISFIES LIE ALGEBRA

$M^{\mu\nu}$ for David Tong →

$$(J^{\mu\nu})_{\underbrace{\alpha\beta}_{4\times4 \text{ matrix}}} = i\left(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha\right)$$

Then a vector field transforms as

$$\begin{aligned} V'^\alpha(x' = \Lambda x) &= (S(\Lambda))^\alpha{}_\beta V^\beta(x) \\ &= (e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}})^\alpha{}_\beta V^\beta(x) \\ &= \left(\delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta + O(\omega^2)\right)V^\beta(x) \\ &= \left(\delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})_{\gamma\beta}\, g^{\gamma\alpha} + O(\omega^2)\right)V^\beta(x) \end{aligned}$$

} infinitesimal transformation

A reminder: $(J^{\mu\nu})_{\alpha\beta}$ are six 4×4 matrices which are antisymmetric in $\mu\nu \Rightarrow J^{01} = -J^{10}$ and in $\alpha\beta \Rightarrow (J^{01})_{10} = -(J^{01})_{01}$

although be careful with e.g. $(J^{\mu\nu})^\alpha{}_\beta$!

$\omega_{\mu\nu}$ are six numbers telling us what kind of transformation to do.

Example

Rotation about the z-axis is given by

$$\omega_{12} = -\omega_{21} = \delta\theta \qquad \omega_{ij} = 0 \ \forall\, i,j \neq \{1,2\}$$

$$\begin{pmatrix} V'^0 \\ V'^1 \\ V'^2 \\ V'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \delta\theta & 0 \\ 0 & -\delta\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

Example

Boost along the x-axis is given by

$$\omega_{01} = -\omega_{10} = \delta\beta$$

$$\begin{pmatrix} V'^0 \\ V'^1 \\ V'^2 \\ V'^3 \end{pmatrix} = \begin{pmatrix} 1 & \delta\beta & 0 & 0 \\ -\delta\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

Spinors

We already knew about the vector representation (after all x^μ is a four-vector). What about spin-½ objects? To answer this we will take another apparent tangent and study the Clifford algebra, which defines a set of matrices that obey

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}}$$

$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$

In other words, the matrices γ^μ obey

$$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \quad \text{for } \mu \neq \nu$$

$$(\gamma^0)^2 = \mathbb{1}$$

$$(\gamma^i)^2 = -\mathbb{1} \qquad i = 1,2,3$$

It turns out there are no 2×2 or 3×3 matrices that satisfy the Clifford algebra. The simplest such matrices are in fact 4×4 matrices.

related by $V\gamma^\mu V^{-1}$

but only one irreducible rep

There are many choices of "basis"

Weyl ("chiral") basis : $\gamma^0 = \begin{pmatrix} 0_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 0_{2\times2} \end{pmatrix}$

$$\gamma^i = \begin{pmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{pmatrix}$$

Dirac basis : $\gamma^0 = \begin{pmatrix} 1_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -1_{2\times2} \end{pmatrix}$

$$\gamma^i = \begin{pmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{pmatrix}$$

Here the σ^i are 2×2 Pauli matrices, which satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$

again, many basis options, one is

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So what do these Clifford algebra matrices have to do with anything

(a) "Weyl basis"

$$\gamma^0 = \begin{pmatrix} 0_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 0_{2\times2} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{pmatrix}$$

There are no unitary representations of the Lorentz group

$$\overset{1/2}{J}{}^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

- generates boosts
- block diagonal
- not Hermitian
- $S(\Lambda_{boost})$ not unitary

$$\overset{1/2}{J}{}^{ij} = \frac{\epsilon_{ijk}}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

- generates rotations
- block diagonal
- Hermitian
- $S(\Lambda_{rot})$ unitary

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 1_{2\times2} \end{pmatrix}$$

- block diagonal

(b) "Dirac basis"

γ^5 intimately associated with chiral symmetry

$$\gamma^0 = \begin{pmatrix} 1_{2\times2} & 0_{2\times2} \\ 0 & -1_{2\times2} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{pmatrix}$$

$$\overset{1/2}{J}{}^{0i} = \frac{i}{2}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

- generates boosts
- not block diagonal
- not Hermitian

$$\overset{1/2}{J}{}^{ij} = \frac{\epsilon_{ijk}}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

- generates rotations
- block diagonal
- Hermitian

$$\gamma^5 = \begin{pmatrix} 0_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 0_{2\times2} \end{pmatrix}$$

- not block diagonal

Aside

The object ϵ_{ijk} is not a tensor – it is a tensor density (or pseudotensor) in flat spacetime.

of weight 1

$$\tilde{\epsilon}_{\mu_1'\cdots\mu_n'} = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| \tilde{\epsilon}_{\mu_1\cdots\mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

In particular it changes sign under improper Lorentz transformations

det = −1

We can define a true tensor in curved spacetime through

$$\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}.$$

square root of the determinant of the metric

In curved spacetime $\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}$ becomes the Levi-Civita symbol, which is defined to be $\tilde{\epsilon}_{12\cdots n} = +1$ (and so on) in any coordinate system.

The commonly defined $\tilde{\epsilon}^{\mu_1\mu_2\cdots\mu_n}$, whose components are numerically equal to those of $\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}$, is a tensor density of weight −1

$$\epsilon^{\mu_1\mu_2\cdots\mu_n} = \frac{\mathrm{sgn}(g)}{\sqrt{|g|}}\, \tilde{\epsilon}^{\mu_1\mu_2\cdots\mu_n}$$

Note: ϵ can only be defined as a tensor on orientable manifolds

can consistently choose "clockwise" loops – or surface normal vector

But what is an improper Lorentz transformation?

To answer this, we need a little more group theory.

The Lorentz group is $O(1,3)$, which preserves $ds^2 = dt^2 - d\bar{x}^2$.

- noncompact – can't "hold it in your hands" (∼ finite and bounded)
  SU(2) is compact
- not connected – not "joined up"
- 6-dimensional – = 6 generators
- non-Abelian Lie group – generators don't commute, Lie groups are also differentiable manifolds and have smooth operations

NOT SUBGROUP

The Lorentz group has four (separate) connected bits,
which are labelled by their properties under parity (P)
and time reversal (T) - either the elements do or
don't change sign under these two operations.

The proper orthochronous Lorentz group, $SO^{+}(1,3)$
contains the identity and all transformations
that have determinant $= +1$ (= proper) and
that preserve the direction of time (orthochronous). This is
the connected piece that contains ordinary rotations and
boosts and is the one we live in (in some sense).

also the "restricted Lorentz group"

Every element in $O(1,3)$ can be written as the semi-direct
product $O(1,3) \simeq SO^{+}(1,3) \otimes \{\mathbb{1}, P, T, PT\}$.

So ϵ_{ijk} can be thought of as a "tensor" within $SO^{+}(1,3)$ but
it is not really a tensor, since it changes sign under P.

OK, back to fermions!

We just introduced the gamma matrices, and the Clifford
algebra, which gave us a spinor representation of the Lorentz
group.

Recall that our fields were called scalar, vector, or tensor
according to whether they transformed under the scalar,
vector, or tensor representation of the Lorentz group.

The same is true of spinors. We are looking for a field for the spinor representation matrices (ie the gamma matrices) to act on. Since the gamma matrices are 4×4 matrices, it is natural to take the corresponding field as a four-component object. We will denote this field $\psi(x)$.

it is a vector, but not in the same space as x^μ, so is not a 4-vector in the same way :)

Now we want to construct an action for our spinor field.

We want our action to be a Lorentz scalar, so what can we build that is a Lorentz scalar?

If we define $\psi^\dagger(x) = (\psi^*)^T(x)$ then we might try $\psi^\dagger(x)\psi(x)$. Does this work?

N.B. $\psi^\dagger\psi \Rightarrow S^T S = 1$ not true in general

represents $\sum_{a=1}^{4} \psi_a^\dagger(x)\,\psi_a(x)$

$$\psi(x) \to S(\Lambda)\psi(x)$$

$$\psi^\dagger(x) \to \psi^\dagger(x)\,(S(\Lambda))^\dagger$$

- N.B. $\psi'(x') = S(\Lambda)\psi(x)$
- $S(\Lambda) = \exp(\tfrac{i}{2}\omega_{\mu\nu}\,{}^{\frac{1}{2}}\mathcal{J}^{\mu\nu})$
- summation over spinor indices!

So

$$\psi^\dagger(x)\psi(x) \to \psi^\dagger(x)(S(\Lambda))^\dagger S(\Lambda)\,\psi(x)$$

But there are no finite unitary representations of the Lorentz group so $(S(\Lambda))^\dagger S(\Lambda) \neq \mathbb{1}$

$S(\Lambda)$ is not unitary for boosts.

requires ${}^{\frac{1}{2}}\mathcal{J}^{\mu\nu}$ to be antihermitian $\Rightarrow$ all γ^μ to be either hermitian or antihermitian

It turns out that what works is

$$\bar{\Psi}(x)\,\Psi(x) \equiv \left(\Psi^+(x)\gamma_0\right)\Psi(x)$$

$$\rightarrow \Psi^+(x)\,S^+(\Lambda)\,\gamma^0 S(\Lambda)\,\Psi(x) = \Psi^+(x)\,\gamma^0\,\Psi(x) = \bar{\Psi}(x)\Psi(x)$$

when you can explicitly show $S^+(\Lambda)\,\gamma^0 S(\Lambda) = \gamma^0$ as follows

$$\left(1 - \frac{i}{2}\delta\omega_{\mu\nu}\,{}^{\frac{1}{2}}\!J^{\mu\nu}\right)^+ \gamma^0 \left(1 - \frac{i}{2}\delta\omega_{\mu\nu}\,{}^{\frac{1}{2}}\!J^{\mu\nu}\right) \leftarrow \text{infinitesimal}$$

$$= \left(1 + \frac{i}{2}\delta\omega_{\mu\nu}\left({}^{\frac{1}{2}}\!J^{\mu\nu}\right)^+\right)\gamma^0\left(1 - \frac{i}{2}\delta\omega_{\mu\nu}\,{}^{\frac{1}{2}}\!J^{\mu\nu}\right)$$

$$= \gamma^0 + \frac{i}{2}\delta\omega_{\mu\nu}\left(\left({}^{\frac{1}{2}}\!J^{\mu\nu}\right)^+\gamma^0 - \gamma^0\,{}^{\frac{1}{2}}\!J^{\mu\nu}\right) + O(\delta\omega^2)$$

Then, for boosts

$$\left({}^{\frac{1}{2}}\!J^{0i}\right)^+ = \left(\frac{i}{4}\left[\gamma^0,\gamma^i\right]\right)^+ = \left(\frac{i}{2}\gamma^0\gamma^i\right)^+ = -\frac{i}{2}(\gamma^i)^+\gamma^{0+} = -\frac{i}{2}(\gamma^i)^+\gamma^0$$

$$= -\frac{i}{2}\gamma^0\gamma^0\gamma^{i+}\gamma^0 \underset{PTO}{=} -\frac{i}{2}\gamma^0\gamma^i = -\,{}^{\frac{1}{2}}\!J^{0i} \qquad \leftarrow \text{antihermitian}$$

$$\Rightarrow \left({}^{\frac{1}{2}}\!J^{0i}\right)^+\gamma^0 - \gamma^0\left({}^{\frac{1}{2}}\!J^{0i}\right) = -\left\{\gamma^0, {}^{\frac{1}{2}}\!J^{0i}\right\} = -\left\{\gamma^0, \frac{i}{2}\gamma^0\gamma^i\right\} = -\frac{i}{2}\left(\gamma^0\gamma^0\gamma^i + \gamma^0\gamma^i\gamma^0\right)$$

$$= -\frac{i}{2}\left(\gamma^i - \gamma^i\right) = 0 \qquad \Rightarrow \quad S^+(\Lambda)\,\gamma^0 S(\Lambda) = \gamma^0 \quad \checkmark$$

and, for rotations

$$\left({}^{\frac{1}{2}}\!J^{ij}\right)^+ \overset{i\neq j}{=} \left(\frac{i}{4}\left[\gamma^i,\gamma^j\right]\right)^+ = \left(\frac{i}{2}\gamma^i\gamma^j\right)^+ = -\frac{i}{2}\gamma^{j+}\gamma^{i+} = -\frac{i}{2}\gamma^0\gamma^j\gamma^0\gamma^0\gamma^i\gamma^0$$

$$= -\frac{i}{2}\gamma^0\gamma^j\gamma^i\gamma^0 = -\frac{i}{2}\gamma^j\gamma^i = \frac{i}{2}\gamma^i\gamma^j = {}^{\frac{1}{2}}\!J^{ij} \qquad \leftarrow \text{hermitian}$$

$$\Rightarrow S^+(\Lambda_{rot}) = S^{-1}(\Lambda_{rot})$$

$$\Rightarrow S^+(\Lambda)\,\gamma^0 S(\Lambda) = S^{-1}(\Lambda)\,\gamma^0 S(\Lambda) = \Lambda^0{}_\nu\,\gamma^\nu \underset{\uparrow}{=} \Lambda^0{}_0\,\gamma^0 = \gamma^0$$

since $\Lambda^0{}_0 = 1$ is only nonzero $\Lambda^0{}_\nu$ for rotations.

It turns out that

- $\bar{\Psi}(x)\gamma^{\mu}\Psi(x)$ transforms as a Lorentz vector
  $\Rightarrow$ we can construct Lorentz scalars by contracting the free index

| Proofs: David Tong's notes p. 88-89

- $\bar{\Psi}(x)\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n}\Psi(x)$ transforms as a Lorentz tensor

So the scalar action we want is built from these components - well, in fact, from the first two:

$$S = \int d^4x \; \bar{\Psi}(x)\left(i\gamma^{\mu}\partial_{\mu} - m\right)\Psi(x)$$

↑ in general could be any scalar A but we will see this gives the right dispersion relation.

The corresponding equations of motion are

varying w.r.t. $\bar{\Psi}$

- $(i\gamma^{\mu}\partial_{\mu} - m)\Psi(x) = 0$ — the <u>Dirac equation</u>
- $i\partial_{\mu}\bar{\Psi}\gamma^{\mu} - m\bar{\Psi} = 0$

| "completely gorgeous" D. Tong

varying w.r.t. Ψ

If we apply $(i\gamma^{\mu}\partial_{\mu} + m)$ we obtain

$$(i\gamma^{\nu}\partial_{\nu} + m)(i\gamma^{\mu}\partial_{\mu} - m)\Psi(x) = 0$$

$$(-\gamma^{\nu}\gamma^{\mu}\partial_{\nu}\partial_{\mu} - m^2)\Psi(x) = 0$$

$$(\partial^2 + m^2)\Psi(x) = 0 \quad \Rightarrow \quad p^2 - m^2 = 0$$

see next page

relativistic dispersion relation ü

It is usual to introduce the Feynman "slash" notation

$$\not{a} \equiv \sum_{\mu=0}^{3} \gamma^\mu a_\mu$$

And in general we have

$$\not{a}\not{a} = \gamma^\mu \gamma^\nu a_\mu a_\nu = \frac{1}{2}\gamma^\mu\gamma^\nu a_\mu a_\nu + \frac{1}{2}\gamma^\nu\gamma^\mu a_\nu a_\mu$$

$$= \frac{1}{2}\gamma^\mu\gamma^\nu a_\mu a_\nu + \frac{1}{2}\gamma^\nu\gamma^\mu a_\mu a_\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} a_\mu a_\nu$$

$$= \frac{1}{2} 2 g^{\mu\nu} a_\mu a_\nu = a\cdot a = a^2$$

Then we can write the Dirac equation as

$$\boxed{(i\not{\partial} - m)\psi(x) = 0}$$

The Dirac equation is Lorentz invariant

$$(i\gamma^\mu \partial'_\mu + m)\psi'(x') = (i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu + m) S(\Lambda)\psi(x)$$

$$= S(\Lambda) S^{-1}(\Lambda)(i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu + m) S(\Lambda)\psi(x)$$

$$= S(\Lambda)(i S^{-1}(\Lambda)\gamma^\mu S(\Lambda)(\Lambda^{-1})^\nu{}_\mu \partial_\nu + m\, S^{-1}(\Lambda)S(\Lambda))\psi(x)$$

$$= S(\Lambda)(i\Lambda^\mu{}_\rho \gamma^\rho (\Lambda^{-1})^\nu{}_\mu \partial_\nu + m)\psi(x)$$

$$= S(\Lambda)(i\gamma^\rho \delta^\nu{}_\rho \partial_\nu + m)\psi(x)$$

$$= S(\Lambda)(i\gamma^\nu \partial_\nu + m)\psi(x)$$

$$= S(\Lambda)\cdot 0$$

$$= 0 \quad \checkmark$$

The Lorentz group is the set of elements Λ that leave

$x_\mu x^\mu = x^2$ invariant.

If $x' = \Lambda x$ then $x^2 = x'^2 \Rightarrow \eta = \Lambda^T \eta \Lambda \Rightarrow \det \Lambda = \pm 1$.

Taking $\eta = \begin{pmatrix} -1 & 0 \\ 0 & \bar{\bar{1}} \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} r & -\bar{a}^T \\ \bar{b} & \bar{\bar{M}} \end{pmatrix}$

$\bar{a}$ a 3-vector
$\bar{\bar{m}}$ a 3x3 matrix

In this form we can derive $r^2 = 1 + b^T b \Rightarrow \begin{matrix} r \le -1 \\ r \ge 1 \end{matrix}$

Note that the four components of $O(1,3)$ are not subgroups, because it can be shown that the composition of any two Lorentz transformations is always a proper orthochronous transformation:

Examples of improper transformations include Parity $P = \begin{pmatrix} 1 & 0 \\ 0 & -\bar{\bar{1}} \end{pmatrix}$ and time reversal $T = \begin{pmatrix} -1 & 0 \\ 0 & \bar{\bar{1}} \end{pmatrix}$

Transformations of the form $x' = \Lambda x + c$ are inhomogeneous Lorentz transformations and elements of the Poincaré group, which is a story for another time.

boosts, rotations, and transl
10 D group of isometries
of Minkowski spacetime

Before we discuss the free particle solutions of the Dirac equation, we can deduce a couple of interesting properties.

1)
In the Weyl basis

$$\overset{1/2}{J}{}^{0i} = -\frac{i}{2}\begin{bmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{bmatrix} \qquad \overset{1/2}{J}{}^{ij} = \frac{1}{2}\epsilon_{ijk}\begin{bmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{bmatrix}$$

the generators are block-diagonal. The Weyl basis shows the spinor representation is <u>reducible</u> and we can separate the "upper" and "lower" components of ψ separately.

We define

"<u>Weyl spinors</u>"

$$\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \qquad \text{where} \quad \psi_{L,R} = \frac{1}{2}(1 \mp \gamma^5)\psi.$$

in Weyl basis.

$$= \frac{1}{2}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \mp \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)\psi$$

$$= \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\psi \quad \text{or} \quad \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\psi$$

N.B. $P_L^2 = P_R^2 = 1$, $P_L P_R = 0$, $(\gamma^5)^2 = 1$, $\{\gamma^5, \gamma^\mu\} = 0$

N.B. Can define LH/RH spinors in any basis through $P_{L,R}\psi$

Note

$$\psi_L^+ = \psi^+ \tfrac{1}{2}(1 - \gamma^{5+}) = \psi^+ \tfrac{1}{2}(1 - \gamma^5)$$

$$\Rightarrow \bar{\psi}_L = \psi_L^+\gamma^0 = \psi^+\tfrac{1}{2}(1-\gamma^5)\gamma^0 = \psi^+\gamma^0\tfrac{1}{2}(1+\gamma^5) = \bar{\psi}\tfrac{1}{2}(1+\gamma^5)$$

Similarly $\bar{\psi}_R = \bar{\psi}\,\frac{1}{2}(1-\gamma^5)$

So we have

strictly should be $\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$

$$\bar{\psi}\gamma^\mu\psi = (\bar{\psi}_L + \bar{\psi}_R)\gamma^\mu(\psi_L + \psi_R)$$

$$= \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}_L\gamma^\mu\psi_R + \bar{\psi}_R\gamma^\mu\psi_L$$

$$= \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R + \frac{1}{4}\bar{\psi}\left((1+\gamma^5)\gamma^\mu(1+\gamma^5) + (1-\gamma^5)\gamma^\mu(1-\gamma^5)\right)\psi$$

$$= \bar{\psi}_{L,R}\gamma^\mu{}_{L,R} + \frac{1}{4}\bar{\psi}\left(2\gamma^\mu + \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 + \gamma^5\gamma^\mu\gamma^5 - \gamma^5\gamma^\mu - \gamma^\mu\gamma^5 + \gamma^5\gamma^\mu\gamma^5\right)\psi$$

$$= \bar{\psi}_{L,R}\gamma^\mu\psi_{L,R} + \frac{1}{4}\bar{\psi}\cdot 2(\gamma^\mu + \gamma^5\gamma^\mu\gamma^5)\psi$$

$$= \bar{\psi}_{L,R}\gamma^\mu\psi_{L,R} + \frac{1}{2}\bar{\psi}(\gamma^\mu - \gamma^5\gamma^5\gamma^\mu)\psi$$

$$= \bar{\psi}_{L,R}\gamma^\mu\psi_{L,R}.$$

On the other hand

$$\bar{\Psi}\Psi = (\bar{\Psi}_R \bar{\Psi}_L)\begin{pmatrix}\Psi_L\\ \Psi_R\end{pmatrix}$$

$$= \bar{\Psi}_R \Psi_L + \bar{\Psi}_L \Psi_R$$

N.B. We also have the "Weyl equations"

$$i\bar{\sigma}^\mu \partial_\mu \Psi_L = 0$$
$$i\sigma^\mu \partial_\mu \Psi_R = 0$$

So the Lagrangian (density)

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi$$
$$= \bar{\Psi}_L i\not{\partial}\Psi_L + \bar{\Psi}_R i\not{\partial}\Psi_R - m\bar{\Psi}_L\Psi_R - m\bar{\Psi}_R\Psi_L$$

separates Ψ_L and Ψ_R only when $m = 0$.

we will see that we have chiral symmetry for massless fermions.

2)
In the Dirac basis, if we write

$$\Psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip.x}\,\tilde{\Psi}(p)$$

then the Dirac equation becomes $(\not{p} - m)\Psi(p) = 0$.

In the rest frame we have $p^\mu = (m, \vec{0})$, so the Dirac equation becomes $m(\gamma^0 - 1)\tilde{\Psi}(p) = 0$, or:

$$\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix} - \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\tilde{\Psi}(p) = \begin{pmatrix}0&0\\0&2\end{pmatrix}\tilde{\Psi}(p) = 0$$

Dirac basis!

If we further write $\tilde{\Psi}(p) = \begin{pmatrix}\tilde{\phi}(p)\\ \tilde{\chi}(p)\end{pmatrix}$ then the Dirac equation implies

$$0.\tilde{\phi}(p) = 0$$
$$2.\tilde{\chi}(p) = 0$$

$$\tilde{\Psi}(p) = \begin{pmatrix}\tilde{\phi}(p)\\ 0\end{pmatrix}.$$

The Dirac field has two degrees of freedom, which is the right number for a spin-$\frac{1}{2}$ object.