Scalar fields: a summary

Lagrangian: $\mathcal{L} = \underbrace{\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right)}_{\text{free scalar fields}} - \underbrace{\frac{\lambda}{4!}\phi^4}_{\text{interacting fields}}$

Free fields

Satisfy Klein-Gordon equation $(\partial^2 - m^2)\phi = 0$

with solution $\phi(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[e^{-ik\cdot x}a(\bar{k}) + e^{ikx}a^\dagger(\bar{k})\right]$

and dispersion relation $E^2 = \bar{k}^2 + m^2$

Hamiltonian: $\mathcal{H} = \frac{1}{2}\left(\pi^2 + \bar{\nabla}\phi\cdot\bar{\nabla}\phi + m^2\phi^2\right)$

$$H = \int d^3\bar{x}\,\mathcal{H} = \int\frac{d^3\bar{k}}{(2\pi)^3}\,\omega_k\, a^\dagger(\bar{k})\,a(\bar{k})$$

normal ordered

Momentum operator: $P^i = -\int d^3\bar{x}\,\pi\,\nabla^i\phi = \int\frac{d^3\bar{k}}{(2\pi)^3}k^i a^\dagger(\bar{k})a(\bar{k})$

Number operator: $N = \int\frac{d^3\bar{k}}{(2\pi)^3}a^\dagger(\bar{k})a(\bar{k})$

Equal-time commutation relations

$$\left.\begin{aligned}\left[\phi(\bar{x},t),\pi(\bar{y},t)\right] &= i\,\delta^{(3)}(\bar{x}-\bar{y})\\ \left[a(\bar{k}),a^\dagger(\bar{p})\right] &= (2\pi)^3\,\delta^{(3)}(\bar{k}-\bar{p})\end{aligned}\right\}\ \text{all others zero}$$

Propagators:

- Klein-Gordon $$D_{KG}(x-y) = \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip\cdot(x-y)}$$

time ordering

- Feynman $$D_F(x-y) = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

- Unnamed $$\Delta(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d\,p}{(2\pi)} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

acausal

all are Green's functions of the Klein-Gordon operator

Interacting fields

$$\phi_H(\bar{x},t) = U_I^+(t,0)\, \Phi_I(\bar{x},t)\, U_I(t,0)$$

field in Heisenberg picture

field in interaction picture

obeys free field solution $$\phi_I(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(\bar{p})e^{-ip\cdot x} + a^+(p)e^{ip\cdot}\right.$$

time evolution operator

$$U(t,t') = T\left\{\exp\left(-i\int_{t'}^{t} d\tau\, H_I(\tau)\right)\right\}$$

Time-ordered products of fields in Heisenberg picture

$$\langle\Omega|T\{\phi_H(x_1)\phi_H(x_2)\ldots\phi_H(x_n)\}|\Omega\rangle$$
$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n)e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|\ \rangle}{\langle 0|T\{e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}$$

Wick's theorem:

$$T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\} = :\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n): + \text{all possible contractions}$$

Combining these last two points:

$$\langle\Omega|T\{\phi_H(x_1)\phi_H(x_2)\cdots\phi_H(x_n)\}|\Omega\rangle$$

= sum over all connected diagrams with n external spacetime points $\{x_1, x_2, \ldots, x_n\}$

Scattering

this is a theorist's definition

Cross-section : $\sigma = \dfrac{\text{number of events}}{\left(\text{beam particles per unit area}\right)\left(\text{number of target particles}\right)}$

Differential cross-section $\quad \sigma = \displaystyle\int \frac{d\sigma}{d\Omega}\, d\Omega$

Decay rate : $\Gamma = \dfrac{\text{number of decays per unit time}}{\text{number of particles present}}$

Invariant matrix element

$$\langle f|S|i\rangle = (2\pi)^4 \delta^{(4)}(\Sigma p)\, i\mathcal{M}(p) \qquad \text{for } |i\rangle \neq |f\rangle$$

Then

$$d\sigma = \frac{1}{2E_A E_B |v_A - v_B|}\left(\prod_f \frac{d^3\bar{p}_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}(k_A, k_B \to \{p_f\})|^2 \times (2\pi)^4 \delta^{(4)}\Big(k_A + k_B - \sum_f p_f\Big)$$

$$d\Gamma = \frac{1}{2m_A}\left(\prod_f \frac{d^3\bar{p}_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}(m_A \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}\Big(k_A - \sum_f p_f\Big)$$

LSZ reduction

$$\langle f|S|i\rangle = \prod_{i,f}\left[i\int d^4x\, e^{ip_f\cdot x_f}\left(\nabla_f^2+m^2\right)\right]\left[i\int d^4x\, e^{-ip_i\cdot x_i}\left(\nabla_i^2+m^2\right)\right]$$
$$\times \langle\Omega|T\{\phi(x_i)\ldots\phi(x_f)\}|\Omega\rangle$$