Wick's theorem

We have reduced our problem to calculating objects like

$$\langle 0 | T\{\phi_I(x)\, \phi_I(x_2) \dots \phi(x_n)\} | 0 \rangle$$

Q: Why didn't I include H_I here?

This is great! We already know what the answer is for $n=2$

$$\langle 0 | T\{\phi_I(x_1)\, \phi_I(x_2)\} | 0 \rangle = \langle 0 | T\{\phi^{(FREE)}(x_1)\, \phi^{(FREE)}(x_2)\} | 0 \rangle$$

$$= D_F(x_1 - x_2)$$

the Feynman propagator

What about $n > 2$?

Well we know how to write the interaction picture field in terms of the annihilation and creation operators of the free theory. So we could just plug those in and work it all out. But there's an easier way...

To understand this procedure, let's look again at the $n=2$ case

Recall
$$\phi_I(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(\bar{p}) e^{-ip\cdot x} + a^\dagger(\bar{p}) e^{ip\cdot x} \right)$$

$$= \underbrace{\int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a(\bar{p}) e^{-ip\cdot x}}_{\phi_I^+(x)} + \underbrace{\int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a^\dagger(\bar{p}) e^{ip\cdot x}}_{\phi_I^-(x)}$$

If we take $x^0 > y^0$ then

$$T\{\phi_I(x)\phi_I(y)\} = \phi_I(x)\phi_I(y)$$

$$= (\phi_I^+(x) + \phi_I^-(x))(\phi_I^+(y) + \phi_I^-(y))$$

$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^-(y)$$

normal ordering $\rightarrow$
$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + [\phi_I^+(x), \phi_I^-(y)] + \phi_I^-(x)\phi \dots$$

Thus

recall $D(x-y) = [\phi(x),\phi(y)]$ in the free theory $= [\phi_I^+(x),\phi_I^-(y)]$

$$T\{\phi_I(x)\phi_I(y)\} \underset{x^0>y^0}{=} :\phi_I(x)\phi_I(y): + D(x-y)$$

since all terms are normal ordered

If we repeat the argument for $y^0 > x^0$ we have

$$T\{\phi_I(x)\phi_I(y)\} \underset{y^0>x^0}{=} :\phi_I(x)\phi_I(y): + D(y-x)$$

Putting these together, we have

$$T\{\phi_I(x)\phi_I(y)\} = :\phi_I(x)\phi_I(y): + \Delta_F(x-y)$$

Feynman propagator

$$\Delta(x-y) = \theta(x^0-y^0)[\phi_I^+(x),\phi_I^-(y)] + \theta(y^0-x^0)[\phi_I^+(y),\phi_I^-(x)]$$

We can introduce some new notation here

> contraction: replace a pair of fields in a string of fields by the Feynman propagator, leaving all others untouched

denote by $\sqcap$

<u>Examples</u>

$$\overbrace{\phi_I(x)\phi_I(y)} = \Delta_F(x-y)$$

$$\phi_I(x)\overbrace{\phi_I(y)\phi_I(z)} = \phi_I(x)\Delta_F(x-y)$$

With this new notation we can write

$$T\{\phi_I(x)\phi_I(y)\} = :\phi_I(x)\phi_I(y): + \overbrace{\phi_I(x)\phi_I(y)}$$
$$= :\phi_I(x)\phi_I(y) + \overbrace{\phi_I(x)\phi_I(y)}:$$

We can bring "inside" normal ordering symbol because Δ_F is a c-number

$T\{\phi(x)\phi(y)\}$ and $:\phi(x)\phi(y):$ are <u>operators</u> but the difference between them is a c-number!

This is our first example of <u>Wick's Theorem</u>

Proof is by induct
P+S, p. 90

Wick's Theorem: For any string of fields

$$T\{\phi_I(x_1)\,\phi_I(x_2)\cdots\phi_I(x_n)\} = :\phi_I(x_1)\,\phi_I(x_2)\cdots\phi_I(x_n) + \text{all possible pair contractions}:$$

This is particularly useful, because we (generally) want to calculate vacuum matrix elements and for those the normal-ordered term will vanish.

<u>Example</u>

$$\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle = \cancelto{0}{\langle 0|:\phi_I(x)\phi_I(y):|0\rangle} + \langle 0|\Delta_F(x-y)|0\rangle$$
$$= \Delta_F(x-y)\langle 0|0\rangle$$
$$= \Delta_F(x-y)$$

<u>Example</u>

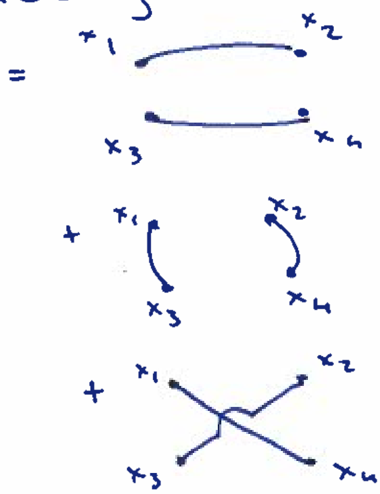
$$
\begin{aligned}
T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\} = \;&:\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4) \\
&+ \wick{\c1\phi_I(x_1)\c1\phi_I(x_2)\,\phi_I(x_3)\,\phi_I(x_4)} \\
&+ \wick{\c1\phi_I(x_1)\phi_I(x_2)\c1\phi_I(x_3)\,\phi_I(x_4)} \\
&+ \wick{\c1\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\c1\phi_I(x_4)} \\
&+ \wick{\phi_I(x_1)\c1\phi_I(x_2)\c1\phi_I(x_3)\,\phi_I(x_4)} \\
&+ \wick{\phi_I(x_1)\c1\phi_I(x_2)\phi_I(x_3)\c1\phi_I(x_4)} \\
&+ \wick{\phi_I(x_1)\,\phi_I(x_2)\c1\phi_I(x_3)\c1\phi_I(x_4)} \\
&+ \wick{\c1\phi_I(x_1)\c1\phi_I(x_2)\c2\phi_I(x_3)\c2\phi_I(x_4)} \\
&+ \wick{\c1\phi_I(x_1)\c2\phi_I(x_2)\c1\phi_I(x_3)\c2\phi_I(x_4)} \\
&+ \wick{\c1\phi_I(x_1)\c2\phi_I(x_2)\c2\phi_I(x_3)\c1\phi_I(x_4)}\,:
\end{aligned}
$$

$$
\begin{aligned}
= \; & :\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4): \\
& + \Delta_F(x_1-x_2):\phi_I(x_3)\phi_I(x_4): + \Delta_F(x_1-x_3):\phi_I(x_2)\phi_I(x_4): \\
& + \Delta_F(x_1-x_4):\phi_I(x_2)\phi_I(x_3): + \Delta_F(x_2-x_3):\phi_I(x_1)\phi_I(x_4): \\
& + \Delta_F(x_2-x_4):\phi_I(x_1)\phi_I(x_3): + \Delta_F(x_3-x_4):\phi_I(x_1)\phi_I(x_2): \\
& + \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) \\
& + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)
\end{aligned}
$$

So the vacuum matrix element is just

$$
\begin{aligned}
\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\}|0\rangle = \; & \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) \\
& + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) \\
& + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)
\end{aligned}
$$

which we can also represent graphically



Wick's theorem is exactly what we need to evaluate arbitrary correlation functions of the form

$$
\langle 0|T\{\phi_1\phi_2\cdots\phi_n \, e^{-i\int_{-T}^{T} d\tau\, H_I(\tau)}\}|0\rangle
$$

$\phi_i \equiv \phi(x_i)$

To do this, we expand the exponential and use Wick's theorem. Let's just take $n=2$ for now. Then

$$\langle 0|T\{\phi_I(x)\phi_I(y)\, e^{-i\int dt\, H_I(t)}\}|0\rangle$$

$$= \underbrace{\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle}_{D_F(x-y)} + (-i)\langle 0|T\{\phi_I(x)\phi_I(y)\int dt\, H_I(t)\}|0\rangle + \dots$$

$$= D_F(x-y) - i\frac{\lambda}{4!}\langle 0|T\{\phi_I(x)\phi_I(y)\int d^4z\, \phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z)\}|0\rangle$$

$$+ O(\lambda^2) \qquad \uparrow \quad \int dt\, H_I = \int dt\int d^3z\, \mathcal{H}_I = \int d^4z\, \mathcal{H}_I$$

Let's look at the second term ($O(\lambda)$). We keep all pairwise contractions.

$\Rightarrow$ if $\underbrace{\phi_I(x)\phi_I(y)}_{= D_F(x-y)}$ $\times$ $\left\{\begin{array}{l} \phi_I(z)\phi_I(z)\;\phi_I(z)\phi_I(z) \\ \phi_I(z)\phi_I(z)\;\phi_I(z)\phi_I(z) \\ \phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z) \end{array}\right\}$ = 3 identical objects $D_F(z-z)D_F(z-z)$

$\Rightarrow$ if $\phi_I(x)\,\phi_I(y)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z) \rightarrow \phi_y\phi_z\,\phi_z\phi_z,\ \phi_y\phi_z\phi_z\phi_z,\ \phi_y\phi_z\phi_z\phi_z$

$\phi_I(x)\phi_I(y)\phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z) \rightarrow$ 3 more options

$\phi_I(x)\phi_I(y)\phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z) \rightarrow$ 3 more options

$\phi_I(x)\phi_I(y)\phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z) \rightarrow$ 3 more options

12 identical objects
$D_F(x-z)D_F(y-z)\,D_F(z-z)$

This is all the possibilities!

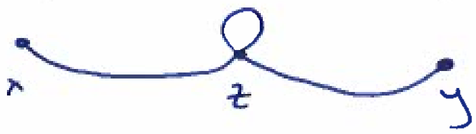
So we can write

$$\langle 0 | T\{\phi_I(x)\phi_I(y)(-i)\frac{\lambda}{4!}\int d^4z\, \phi_I^4(z)\}|0\rangle$$

$$= 3\cdot(-i)\frac{\lambda}{4!} D_F(x-y)\int d^4z\, D_F(z-z)D_F(z-z)$$

$$+ 12\cdot(-i)\frac{\lambda}{4!}\int d^4z\, D_F(x-y)D_F(y-z)D_F(z-z)$$

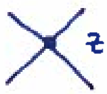
$\cong$ 

$+$ 

In fact we can use this diagrammatic notation to express and correlation function. Our example of $\lambda\phi^4$ theory has two graphical elements:

<u>propagators</u> 

$= D_F(x-y)$

<u>vertices</u> 

$= -i\lambda\int d^4z$ Q: why no factor of 4

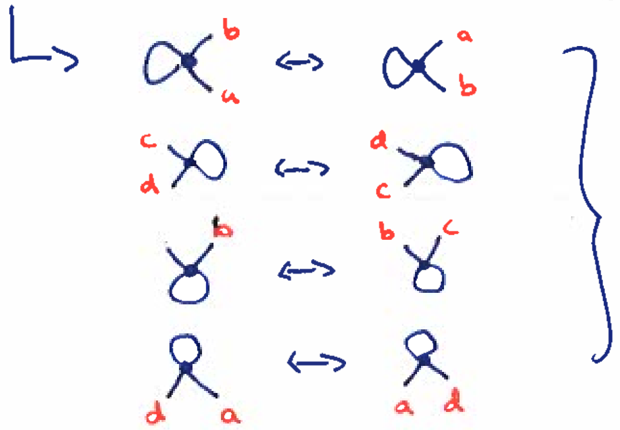
and <u>symmetry factors</u>

$\boxed{\text{Symmetry factor: divide by the number of permutations of internal elements that leave the diagram unchanged}}$

In the example we considered we have two diagrams

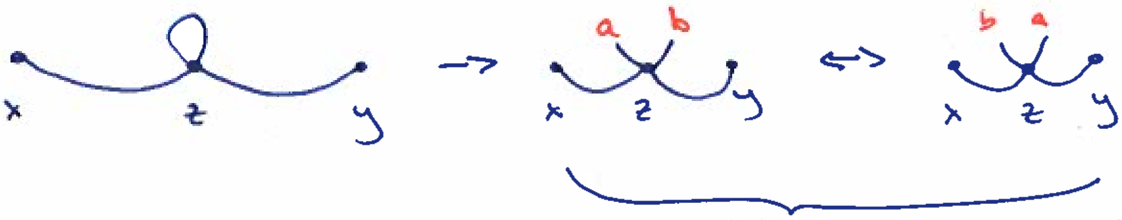
$\downarrow$

no symmetry factor



8 possibilities
$\Rightarrow$ symmetry factor is $1/8$

N.B $\frac{3}{4!} = 1/8$ ✓



2 possibilities $\Rightarrow$ symmetry factor $= 1/2$

N.B. $\frac{12}{4!} = \frac{1}{2}$ ✓

Now we see why we chose a factor of $4!$ - it simplifies our symmetry factors and our position space Feynman rule is just $(-i\lambda)$.

Q: What happens with ϕ^3 theory?

We have just said that our choice of $1/4!$ was dictated by the X vertex. Beyond leading order, this comes from the <u>four-point function</u>. Let's look at ie λ^0 that in more detail.

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\, e^{-i\int dt\, H_I(t)}\}|0\rangle}{\langle 0|T\{e^{-i\int dt\, H_I(t)}\}|0\rangle}$$

$$= \langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle$$

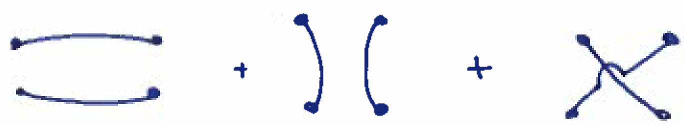
$$+ (-i)\frac{\lambda}{4!}\int d^4z\, \langle 0|T\{\phi_1\phi_2\phi_3\phi_4\,\phi_z\phi_z\phi_z\phi_z\}|0\rangle$$

$$+ \frac{1}{2!}(-i)^2\frac{\lambda^2}{(4!)^2}\int d^4z_1\, d^4z_2\, \langle 0|T\{\phi_1\phi_2\phi_3\phi_4\,\phi_{z1}\phi_{z1}\phi_{z1}\phi_{z1} \times \phi_{z2}\phi_{z2}\phi_{z2}\phi_{z2}\}|0\rangle$$

$$+ O(\lambda^3)$$

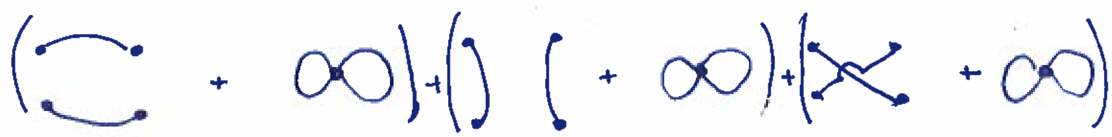
Order-by-order this is

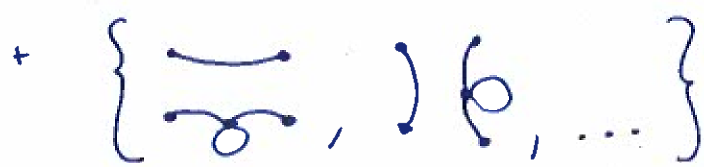
• $O(\lambda^0)$ =

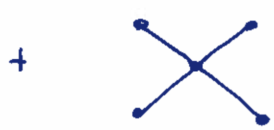


Recall last lecture

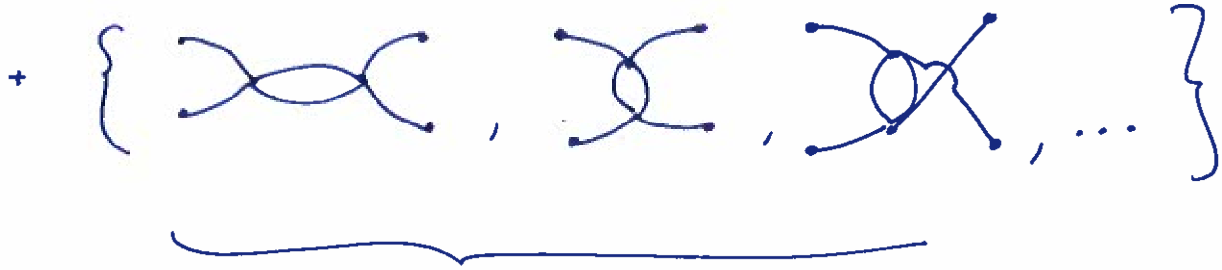
• $O(\lambda^1)$ =







• $O(\lambda^2)$ =



these diagrams are called "connected" and "undressed"

ne-particle Irreducible: cutting one ...opagator does ...t cause the ...iagram to fall apart.

connected: all propagators are continuously connected to at least one external spacetime point.

undressed: all propagators are just simple propagators, with none that look like 

All of this has been done in position space. In most cases it is easier to work directly in momentum space. For that we need the (momentum space) Feynman rules.

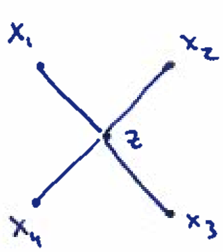
From our integral representation of the Feynman propagator,

$$D_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

we can deduce

$$= \tilde{D}_F(p) = \frac{i}{p^2-m^2+i\epsilon}$$

And the vertex is given by



$$= -i\lambda \int d^4z \, D_F(x_1-z) D_F(x_2-z) D_F(x_3-z) D_F(x_4-z)$$

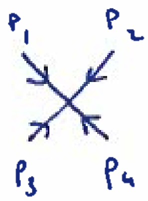
$$= -i\lambda \int d^4z \int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4p_4}{(2\pi)^4}$$

$$\times \frac{ie^{-ip_1\cdot(x_1-z)}}{p_1^2-m^2+i\epsilon}\cdot\frac{ie^{-ip_2\cdot(x_2-z)}}{p_2^2-m^2+i\epsilon}\cdot\frac{ie^{-ip_3\cdot(x_3-z)}}{p_3^2-m^2+i\epsilon}\cdot\frac{ie^{-ip_4\cdot(x_4-z)}}{p_4^2-m^2+i\epsilon}$$

$$= -i\lambda \int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4p_4}{(2\pi)^4}\,\frac{ie^{-ip_1\cdot x_1}}{p_1^2-m^2+i\epsilon}\,\frac{ie^{-ip_2\cdot x_2}}{p_2^2-m^2+i\epsilon}\,\frac{ie^{-ip_3\cdot x_3}}{p_3^2-m^2+i\epsilon}\,\frac{ie^{-ip_4\cdot x_4}}{p_4^2-m^2+i\epsilon}$$

$$\times \underbrace{\int d^4z\; e^{i(p_1+p_2+p_3+p_4)\cdot z}}_{(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)}$$

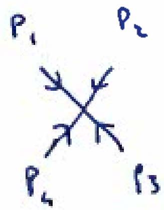
$(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)$ <=> represents momentum conservation at the vertex

=> 

$= -i\lambda\,(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)$

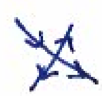
So our Feynman rules are:

- propagator $\xrightarrow{p} = \dfrac{i}{p^2-m^2+i\epsilon}$

- vertex 

$= -i\lambda\,(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)$

direction is somewhat arbitrary but fixed by delta function

 $= -i\lambda(2\pi)^4\delta^{(4)}(p_1-p_2+p_3+p_4)$

- external point $\xrightarrow{p}\bullet\, x = e^{-ip\cdot x}$

N.B. $\xleftarrow{p}\bullet\, x = e^{+ip\cdot x}$

- integrate over undetermined momenta
- divide by symmetry factor

We're looking good – but there's an important piece we've forgotten – the denominator!

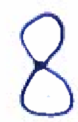
Recall our formula was

$$\langle \Omega | \phi^H(x_1)\,\phi^H(x_2) \ldots \phi^H(x_n) | \Omega \rangle = \lim_{T \to \infty(1-i\epsilon)} \frac{\langle 0 | T\{\phi_1 \phi_2 \cdots \phi_n e^{-\int_{-T}^{T} d\tau\, H_I(\tau)}\} | 0 \rangle}{\langle 0 | T\{ e^{-\int_{-T}^{T} d\tau\, H_I(\tau)} \} | 0 \rangle}$$

What happens when we expand this denominator?

$$\langle 0 | T\{ e^{-i\int d\tau\, H_I(\tau)} \} | 0 \rangle = \langle 0 | T\left\{ 1 + \frac{(-i\lambda)}{4!} \int d^4z\, \phi^4(z) + O(\lambda^2) \right\} | 0 \rangle$$

$$= \langle 0 | 0 \rangle + \frac{(-i\lambda)}{4!} \int d^4z\, \langle 0 | \phi^4(z) | 0 \rangle + O(\lambda^2)$$

$$= 1 + \frac{(-i\lambda)}{4!}\, 3 \cdot \int d^4z$$  $$+ O(\lambda^2)$$

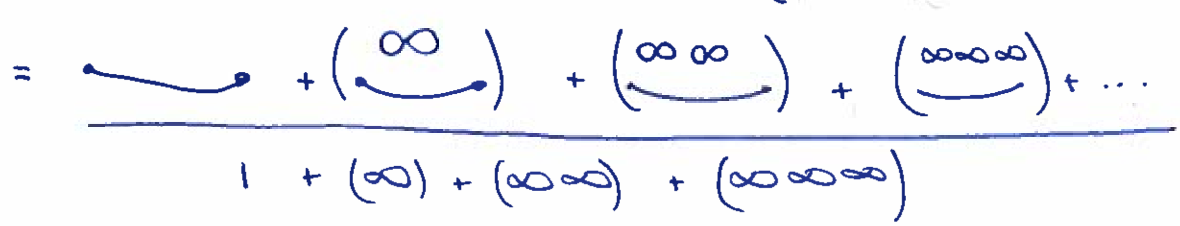
But we've seen such <u>disconnected</u> terms before – in the numerator!

A careful analysis (see p. 95–98 of Peskin + Schroeder) shows that all such disconnected diagrams cancel.

> We interpret these as "vacuum fluctuations" – the interacting vacuum is not "empty"!

Rather than prove this in generality, let's look at our favorite example

$$\langle \Omega | T\{\phi^H(x)\phi^H(y)\} | \Omega \rangle = \frac{\langle 0 | T\{\phi(x)\phi(y)\, e^{-i\int d\tau\, H_I(\tau)}\} | 0 \rangle}{\langle 0 | T\{ e^{-i\int d\tau\, H_I(\tau)} \} | 0 \rangle}$$



$$= D_F(x-y)\, \frac{\left(1 + \infty + \frac{1}{2}(\infty)^2 + \frac{1}{3!}(\infty)^3 + \ldots\right)}{\left(1 + \infty + \frac{1}{2}(\infty)^2 + \frac{1}{3!}(\infty)^3 + \ldots\right)}$$

$$= \frac{D_F(x-y)\, \exp(\infty)}{\exp(\infty)}$$

$$= D_F(x-y)$$

This is not a formal proof! See Peskin + Schroeder.

So, finally, our rule becomes

$$\boxed{\langle\Omega| T\{\phi^H(x_1)\phi^H(x_2)\ldots\phi^H(x_n)\}|\Omega\rangle = \text{sum over all connected diagrams with } n \text{ external points } \{x_1, x_2, \ldots x_n\}}$$

Summary

- Our aim is to "solve" an interacting theory
  - obtain spectrum (ie find energy eigenstates)
  - calculate all possible correlation functions

- To do this we introduced the "interaction picture"

$$\phi_I(\bar{x},t) = e^{iH_0 t}\phi_H(\bar{x},0)e^{-iH_0 t} \qquad (= \phi_0(\bar{x},t))$$

$$\Rightarrow \phi_H(\bar{x},t) = U_I^{+}(t,0)\,\phi_I(\bar{x},0)\,U_I(t,0)$$

$$U_I(t,t') = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} = T\left\{\exp\left(-i\int_{t'}^{t} d\tau\, H_I(\tau)\right)\right\}$$

follows from $i\partial_t U_I = H_I U_I$

- Used the interaction picture to show

$$\langle\Omega|T\{\phi_1^H\ldots\phi_n^H\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_1\ldots\phi_n e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}$$

- Then we used Wick's theorem to relate

$$T\{\phi_1 \dots \phi_n\} = :\phi_1 \dots \phi_n + \text{all possible contractions:}$$

- That allowed us to write

$$\langle \Omega | T\{\phi_1^H \dots \phi_n^H\} | \Omega \rangle = \text{sum over all connected diagrams with external points } x_1 \dots x_n.$$