Interacting scalar fields

So far we're considered *free* fields – the objects we obtain by applying the quantisation condition

$[\phi(\bar{x},t), \pi(\bar{y},t)] = i\delta^{(3)}(\bar{x}-\bar{y})$ to the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$

We showed that the eigenstates of this theory are states of n "particles", each with energy $E_k = \sqrt{|\bar{k}|^2 + m^2}$. But nothing interesting happens to these eigenstates – to make life interesting, we have to add *interactions*. These are terms higher order in the fields

$$\rightarrow \quad \mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - \underbrace{\sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n}_{\mathcal{L}_{INT}}$$

for convenience (will become apparent later)

λ_n are *coupling constants*

Such interaction terms will couple different Fourier modes and allow particles to talk to each other (interact).

What restrictions do we have on possible interactions?

- causality $\Rightarrow$ put all fields at the same spacetime point, so our theory is "local".

- dimensionality $\Rightarrow$ note that $[\mathcal{L}] = 4$ ← follows from $e^{iS} \Rightarrow [S]=0$ also $[x]=-1$, $[\partial_\mu]=1$

  so $[\phi]=1$

  $[m]=1$

  $[\lambda_n] = 4-n$

$\boxed{\text{I will drop operator hats from now on!}}$

this is a one-page summary of a topic that comes up — in QFT 2: effective field theory and the renormalisation group

⇒ three cases:

1) $n < 4$ so $[\lambda_n] > 0$ and the dimensionless combination is $\frac{\lambda_n}{E^{4-n}}$ where E is some mass/energy scale, usually taken to be the scale of the process of interest. Then $\frac{\lambda_n}{E^{4-n}}$ is small at high energies and large at low energies. The coupling is described as <u>relevant</u> because is relevant at low energies, where most stuff happens.

2) $n = 4$ so $[\lambda_n] = 0$ and the coupling is dimensionless. Then $\lambda_n \ll 1$ is the requirement for the application of <u>perturbation theory</u>. Coupling is <u>marginal</u>.

3) $n > 4$ so $[\lambda_n] < 0$ and the dimensionless combination is $\lambda_n E^{n-4}$. This is small at low energies and large at high energies; the coupling is called <u>irrelevant</u>.

• perturbatarity ⇒ our theory is only analytically tractable if the coupling is weak, ie $\lambda \ll 1$.

not a real word

in this course we will always work in the "weak coupling regime"

But strongly coupled theories are really cool:
- bound states + confinement
- charge fractionalisation
- dualities w/ gravity

see Tong p. 50

We should be grateful most couplings are irrelevant – it makes QFT simple. But it also makes it hard to work out whats going on in our GUT up at M...

We will discuss three examples of interacting theories

treated extensively by Peskin + Schroeder

- " ϕ^4 theory"

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 - \frac{\lambda_4}{4!}\phi^4\right)$$

- Scalar Yukawa theory

treated by Tong

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - M_\psi^2 \psi^* \psi + \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m_\phi^2 \phi^2\right) - g\psi^*\psi\phi$$

complex scalar   real scalar $g \ll m_\phi, M_\psi$

- Quantum electrodynamics ... eventually

first we have to understand fermions and gauge symmetr

Let's consider ϕ^4-theory:

- equation of motion now $(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$

can no longer be solved using Fourier decomposition

$\Rightarrow$ to solve this we need new techniques

- "solving" means "diagonalising the Hamiltonian" or finding the energy eigenstates

- We start by first noting there is a new term in the Hamiltonian $H_{INT} = -L_{INT} = \frac{\lambda}{4!}\int d^3\bar{x}\, \phi^4(x)$

Ultimately, what we want to calculate are correlation functions - and finding all n-point functions is another way of "solving" the theory

Interaction picture

> Aim of all of this interaction picture stuff: show that
> $$\langle\Omega|T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_I(x_1)\cdots\phi_I(x_n)e^{-i\int dt\, H_I}\}|0\rangle}{\langle 0|T\{e^{-i\int dt\, H_I}\}|0\rangle}$$
> ↑ eventually used in scattering amplitudes

Recall:

- Schrödinger picture – states evolve – operators don't $\quad i\frac{d}{dt}|\psi\rangle_s = H|\psi\rangle_s$

- Heisenberg picture – operators evolve – states don't $\quad O_H(t) = e^{iHt}O_s e^{-iHt}$, $\quad |\psi\rangle_H = e^{iHt}|\psi\rangle_s$

Introduce:

- Interaction picture – hybrid of the two
  – split $H = H_0 + H_{INT}$

  H_0: controls time dependence of operators

  H_{INT}: controls time dependence of states

> Split into $H_0 + H_{INT}$ is arbitrary but useful if H_0 is exactly solvable

$$\Rightarrow |\psi\rangle_I = e^{iH_0t}|\psi\rangle_s$$
$$O_I(t) = e^{iH_0t}O_s e^{-iH_0t}$$

this applies to H_{INT} so we have
$$H_I \equiv (H_{INT})_I = e^{iH_0t}(H_{INT})_s e^{-iH_0t}$$

then the Schrödinger equation in the interaction picture is
$$i\frac{d}{dt}|\psi\rangle_s = H_s|\psi\rangle_s \Rightarrow i\frac{d}{dt}\left(e^{-iH_0t}|\psi\rangle_I\right) = (H_0 + H_{INT})_s e^{-iH_0t}|\psi\rangle_I$$

chain rule

$$\Rightarrow i\frac{d}{dt}|\psi\rangle_I = e^{iH_0t}(H_{INT})_s e^{-iH_0t}|\psi\rangle_I = H_I|\psi\rangle_I$$

Dyson's formula

This discussion has been formulated in quantum mechanics, but of course we are interested in a field theoretic formulation.

To keep things at least somewhat concrete, we will have in mind the $\lambda\phi^4$ theory, so that

$$H = H_0 + H_{INT} = \underbrace{\int d^3\bar{x}\left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\bar{\nabla}\phi\cdot\bar{\nabla}\phi + \tfrac{m^2}{2}\phi^2\right)}_{H_0} + \overbrace{\frac{\lambda}{4!}\int d^3\bar{x}\,\phi^4}^{H_{INT}}$$

The time dependence of fields is given by ← "Heisenberg" field

$$\phi(\bar{x},t) = e^{iHt}\phi(\bar{x},0)e^{-iHt}$$

$$\Rightarrow \phi(\bar{x},t) = e^{iHt}e^{-iH_0t}\underbrace{e^{iH_0t}\phi(\bar{x},0)e^{-iH_0t}}e^{iH_0t}e^{-iHt}$$

$$\phi_I(\bar{x},t) = e^{iH_0t}\phi(\bar{x},0)e^{-iH_0t}$$

For $\lambda \ll 1$, this gives the dominant time evolution in the theory → time evolution in the free theory!

$$\Rightarrow \phi(\bar{x},t) = U_I^{\dagger}(t,0)\,\phi_I(\bar{x},t)\,U_I(t,0)$$

$$U_I(t,0) \equiv e^{iH_0t}e^{-iHt}$$

The interaction picture field satisfies (by definition)

$$\phi_I(\bar{x},t) = e^{iH_0t}\phi(\bar{x},0)e^{-iH_0t}$$

but clearly $\phi_I(\bar{x},0) = \phi(\bar{x},0)$ so we have

$$\phi_I(\bar{x},t) = e^{iH_0t}\phi_I(\bar{x},0)e^{-iH_0t}$$

Thus the interaction field obeys the same time evolution as the free-field ⇒ we can use our previous decomposition

$$\phi_I(x) = \int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(e^{-ip\cdot x}a(\bar{p}) + e^{ip\cdot x}a^{\dagger}(\bar{p})\right)$$

Now the interaction Hamiltonian, in the interaction picture, is

$$H_I(t) \equiv e^{iH_0t}(H-H_0)e^{-iH_0t}$$

$$= e^{iH_0t}(H_{INT})e^{-iH_0t}$$

make it unwrite → $$= e^{iH_0t}\left(\frac{\lambda}{4!}\int d^3\bar{x}\,\phi^4\right)e^{-iH_0t}$$

$$= \frac{\lambda}{4!}\int d^3\bar{x}\; e^{iH_0t}\phi e^{-iH_0t}e^{iH_0t}\phi e^{-iH_0t}e^{iH_0t}\phi e^{-iH_0t}e^{iH_0t}\phi e^{-iH_0t}$$

$$= \frac{\lambda}{4!}\int d^3\bar{x}\;\phi_I^4$$

So... we have written the time dependence of the "Heisenberg" field in terms of a new "interaction" field and a time evolution operator. Now we need to express that operator in terms of the interaction field.

To do this, we note that

$$i\frac{\partial}{\partial t} U(t,t') = i\frac{\partial}{\partial t}\left(e^{iH_0t} e^{-iH(t-t')} e^{-iH_0t'}\right)$$

$$= e^{iH_0t}(-H_0)e^{-iH(t-t')}e^{-iH_0t'}$$
$$\quad + e^{iH_0t} H e^{-iH(t-t')} e^{-H_0t'}$$
$$= e^{iH_0t}(H-H_0)e^{-iH(t-t')}e^{-iH_0t'}$$
$$= e^{iH_0t}(H-H_0)e^{-iH_0t}e^{iH_0t}e^{-iH(t-t')}e^{-iH_0t'}$$
$$= H_I(t)\,U(t,t')$$

The <u>formal</u> solution to this is given by

$$\boxed{U(t,t') \equiv T\left\{\exp\left(-i\int_{t'}^{t} d\tau\, H_I(\tau)\right)\right\}} \qquad \text{for } U(t,t)=1$$

Before we show this, let's first show how the naive solution

$$U(t,t') \stackrel{?}{=} \exp\left(-i\int_{t'}^{t} d\tau\, H_I(\tau)\right) \qquad \text{fails.}$$

First we have to clarify what this exponential means

$$\exp\left(-i\int_{t'}^{t} d\tau\, H\right) = 1 - i\int_{t'}^{t} H_I\, d\tau + \frac{(-i)^2}{2}\left(\int_{t'}^{t} H_I\, d\tau\right)^2 + \dots$$

Then apply $\partial/\partial t$:

$$\frac{\partial}{\partial t}\exp\left(-i\int_{t'}^{t} d\tau\, H\right) = -iH_I(t) - \frac{1}{2}\left(\int_{t'}^{t} H_I\, d\tau\right)H_I(t) - \frac{1}{2}H_I(t)\left(\int_{t'}^{t} H_I\, d\tau\right)$$

not good, wrong ordering and we can't commute

looks good $\sim H_I U$

So this naive solution does not work – and this is where the time ordering comes in.

Recall $T\{O_1(t_1)\,O_2(t_2)\} = \begin{cases} O_1(t_1)\,O_2(t_2) & t_1 > t_2 \\ O_2(t_2)\,O_1(t_1) & t_2 > t_1 \end{cases}$

Let's now show that the time-ordered exponential <u>does</u> work.

| This is the "Dyson series" approach.

First let's choose, without loss of generality $t' = 0$ and write a formal solution by iteration

$$U_I(t,0) = 1 + (-i)\int_0^t d\tau\, H_I(\tau)\, U_I(\tau,0)$$

$$\Rightarrow U_I(t,0) = 1 + (-i)\int_0^t d\tau_1\, H_I(\tau_1)\left(1 + (-i)\int_0^{\tau_1} d\tau_2\, H_I(\tau_2)\, U_I(\tau_2,0)\right)$$

$$= 1 + (-i)\int_0^t d\tau\, H_I(\tau) + (-i)^2\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2'\, H_I(\tau_1)\, H_I(\tau_2)\, U_I(\tau_2,0)$$

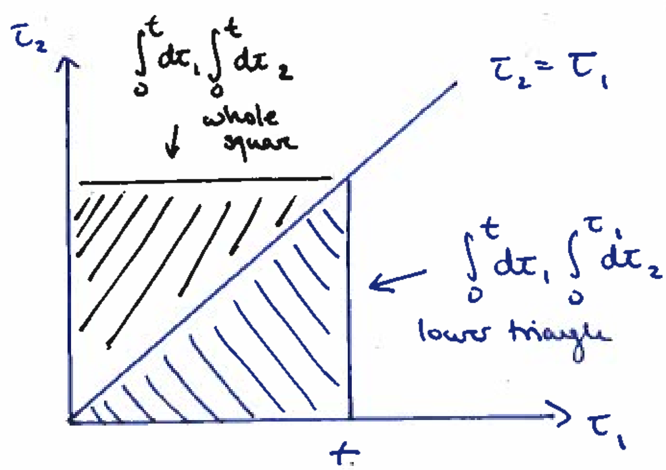
$$\vdots$$

keep repeating this

$$= 1 + (-i)\int_0^t d\tau\, H_I(\tau)$$
$$+ (-i)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2\, H_I(\tau_1)\, H_I(\tau_2)$$
$$+ (-i)^3 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3\, H_1(\tau_1)\, H_I(\tau_2)\, H_I(\tau_3)$$
$$+ \dots$$

Now we notice that

$$\int_0^t d\tau_1 \int_0^{\tau} d\tau_2'\, H_I(\tau_1)\, H_I(\tau_2') = \frac{1}{2}\int_0^t d\tau_1 \int_0^t d\tau_2\, T\{H_I(\tau_1)\, H_I(\tau_2)\}$$



Proof :

$$\int_0^t d\tau_1 \int_0^t d\tau_2 \; T\{H_I(\tau_1) H_I(\tau_2)\} = \underbrace{\int_0^t d\tau_1 \int_0^{\tau_1}}_{\tau_1 > \tau_2} d\tau_2 \; H_I(\tau_1) H_I(\tau_2)$$

$$+ \overbrace{\int_0^t d\tau_2 \int_0^{\tau_2}}^{\tau_2 > \tau_1} d\tau_1 \; H_I(\tau_2) H_I(\tau_1) \quad \text{relabel } \downarrow \tau_1 \leftrightarrow \tau_2$$

$$= \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \; H_I(\tau_1) H_I(\tau_2) + \int_0^t d\tau_1 \int_0^{\tau} d\tau_2 H_I(\tau_1) H_I(\tau_2)$$

$$= 2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau \; H_I(\tau_1) H_I(\tau_2)$$

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In fact one can use the same sort of arguments to show

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n \; H_I(\tau_1) H_I(\tau_2) \ldots H_I(\tau_n)$$

$$= \frac{1}{n!} \int_0^t d\tau_1 \int_0^t d\tau_2 \cdots \int_0^t d\tau_n \; T\{ H_I(\tau_1) H_I(\tau_2) \ldots H_I(\tau_n)\}$$

So our series solution becomes

$$U_I(t,0) = 1 + (-i) \int_0^t d\tau_1 T\{H_I(\tau_1)\}$$

$$+ \frac{(-i)^2}{2!} \int_0^t d\tau_1 \int_0^t d\tau_2 \; T\{H_I(\tau_1) H_I(\tau_2)\}$$

$$+ \frac{(-i)^3}{3!} \int_0^t d\tau_1 \int_0^t d\tau_2 \int_0^t d\tau_3 \; T\{H_I(\tau_1) H_I(\tau_2) H_I(\tau_3)\}$$

$$+$$

$$\equiv T\{e^{-i\int_0^t d\tau H_I(\tau)}\} \qquad \overset{\text{Recall}}{\left[= e^{iH_0 t} e^{-iHt}\right]}$$

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Now we have a (formal) solution for our time-evolution operator, let's look at some of its properties.

First : $U_I(t,t') = T\{\exp(-i \int_{t'}^t d\tau \, H_I(\tau))\}$
$= e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'}$

satisfies : $i \frac{\partial}{\partial t} U_I(t,t') = H_I(t) U_I(t,t')$

} collected here for completeness

Then we also have

$$i\frac{\partial}{\partial t'} U(t,t') = -U_I(t,t')\,H_I(t')$$

To show this we first note

$$U_I^\dagger(t,t') = e^{iH_0t'}\,e^{iH(t-t')}\,e^{-iH_0t} = e^{iH_0t'}\,e^{-iH(t'-t)}\,e^{-iH_0t} = U_I(t',t)$$

then we start from

$$i\frac{\partial}{\partial t} U_I(t,t') = H_I(t)\,U_I(t,t')$$

and relabel $t \leftrightarrow t'$

$$i\frac{\partial}{\partial t'} U_I(t',t) = H_I(t')\,U_I(t',t)$$

$$i\frac{\partial}{\partial t'} U_I^\dagger(t,t') = H_I(t')\,U_I^\dagger(t,t')$$

and dagger this whole equation

$$-i\frac{\partial}{\partial t'} U_I(t,t') = U_I(t,t')\,H_I(t') \qquad \#$$

↓ we use the fact that the Hamiltonian is Hermitian (what happens if it is not?!).

N.B. It is possible to have a non-Hermitian Hamiltonian if your system is PT-symmetric. Unfortunately our Universe is not. See e.g. hep-th/0303005

We can show that time translations work just as we expect

$$U_I(t_1,t_2)\,U_I(t_2,t_3) = e^{iH_0t_1}\,e^{iH(t_1-t_2)}\,e^{-iH_0t_2} \cdot e^{iH_0t_2}\,e^{iH(t_2-t_3)}\,e^{-iH_0t_3}$$

$$= e^{iH_0t_1}\,e^{iH(t_1-t_3)}\,e^{-iH_0t_3}$$

$$= U_I(t_1,t_3)$$

and

$$U_I(t_1,t_3)\,U_I^\dagger(t_2,t_3) = U_I(t_1,t_3)\,U_I(t_3,t_2) = U_I(t_1,t_2).$$

We have now achieved the first of our aims - we have written the Heisenberg field entirely in terms of the interaction field (which, recall, propagates like a free field!).

So far, so good, but also, so what? Well, we're trying to understand correlation functions in the interacting theory and for that we need one more piece - the vacuum state.

The vacuum in the interacting theory, $|\Omega\rangle$, is not the vacuum state in the free theory, $|0\rangle$! $\quad |\Omega\rangle \neq |0\rangle$!

First, $|\Omega\rangle$ is an eigenstate of H, specifically the ground state

$$H|\Omega\rangle = E_\Omega|\Omega\rangle \qquad E_\Omega < E_n \quad \forall\, n>0$$

note $E_\Omega \neq 0$, which is defined by $H_0|0\rangle = 0$

$$H|n_{INT}\rangle = E_n|n_{INT}\rangle$$

take $\langle\Omega|\Omega\rangle = 1$

To isolate this eigenstate, imagine starting with $|0\rangle$ and assume $\langle\Omega|0\rangle \neq 0$ ← otherwise H_{INT} is not a small perturbation!

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$
$$= e^{-iE_\Omega T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle\ |0\rangle$$

Since $E_n > E_\Omega \ \forall\, n>0$, the first term will dominate as $T \to \infty(1-i$

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

comes from $T\to\infty(1-i\epsilon)$ for $e^{-iHT}|0\rangle = e^{-iE_\Omega T}\langle\Omega|0\rangle|$

Similarly we can write

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega T}\langle 0|\Omega\rangle\right)^{-1}\langle 0|e^{-iHT}$$

N.B. <u>same</u> limit

We can write these in a more useful form. Since T is huge, we can shift is by a small constant, t_0

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))}\underbrace{e^{-iH_0(-T-t_0)}|0\rangle}$$

$H_0|0\rangle = 0$
$\Rightarrow e^{-iH_0 t}|0\rangle = |0\rangle$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} U_I(t_0,-T)|0\rangle$$

Similarly

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

So, up to a multiplicative complex constant, we can obtain the interacting theory vacuum state by time evolving the free theory vacuum state!

We are now in a position to start studying correlation functions. Let's make it concrete by considering $\lambda\phi^4$ theory and the two point function $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$.

To make our lives simpler, take $x^0 > y^0 > t_0$

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}\langle 0|U_I(T,t_0)$$

$$\times\left(U_I(x^0,t_0)\right)^{\dagger}\phi_I(x)\,U(x^0,t_0)$$

$$\times\left(U_I(y^0,t_0)\right)^{\dagger}\phi_I(y)\,U(y^0,t_0)$$

$$\times\, U_I(t_0,-T)|0\rangle\left(e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-2iE_\Omega T}\right)$$

$$\times\langle 0|U(T,x^0)\,\phi_I(x)\,U_I(x^0,y^0)\,\phi_I(y)\,U_I(y^0,-T)|0\rangle$$

This looks pretty promising, except for that pesky factor out front. We can deal with that by using $\langle\Omega|\Omega\rangle = 1$

$$\Rightarrow 1 = \langle\Omega|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \langle 0|U_I(T,t_0)\left(e^{-iE_\Omega(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

$$\times \left(e^{-iE_\Omega(t_0+T)}\langle\Omega|0\rangle\right)^{-1} U_I(t_0,-T)|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-i2E_\Omega T}\right)^{-1}\langle 0|U_I(T,t_0)U_I(t_0,-T)|0\rangle$$

Multiplying our expression by unity in this form gives us

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle \underset{x^0>y^0}{=} \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U_I(T,x^0)\,\phi_I(x)\,U_I(x^0,y^0)\,\phi_I(y)U_I(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

This equation also holds if $y^0 > x^0$, because the fields on both sides are time ordered.

$$\Rightarrow \langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,-T)|0\rangle^{-1}$$

$$\times \langle 0|T\{U_I(T,x^0)\,\phi_I(x)\,U_I(x^0,y^0)\,\phi_I(y)\,U_I(y^0,-T)\}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,-T)|0\rangle^{-1}$$

$$\times \langle 0|T\{\phi_I(x)\,\phi_I(y)\,U_I(T,x^0)\,U_I(x^0,y^0)\,U_I(y^0,-T)\}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,-T)|0\rangle^{-1}$$

$$\times \langle 0|T\{\phi_I(x)\,\phi_I(y)\;U_I(T,-T)\}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x)\,\phi_I(y)\,e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}{\langle\; T\{e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}$$

The generalisation to an arbitrary number of fields follows the same logic and leads us to

$$\boxed{\begin{aligned}&\langle\Omega|T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\}|\Omega\rangle\\&=\lim_{T\to\infty(1-i\epsilon)}\frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\dots\phi_I(x_n)\,e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}\end{aligned}}$$

That's it! We have written time-ordered correlation functions entirely in terms of correlation functions of (time-ordered) fields in the interaction picture – and we know how those behave: they evolve in time just as the free fields do.

Some comments:

- this expression is exact, but is most useful when H_I is small (ie $\lambda \ll 1$) and we can approximate the exponential well with just a few (or one) term(s).
- $U_I(\infty,-\infty)$ is called the "scattering matrix" or "S-matrix"

Really it should be "scattering operator"

Lots of time was spent in the mid 20th century trying to solve QFTs through the unitarity of the S-matrix, but this is no longer a focus of most QFT.