

To check our interpretation, we need an operator that measures momentum. Recall the (classical) energy-momentum tensor

$$T^{\mu}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\nu}\phi - \mathcal{L} \delta^{\mu}_{\nu} \quad \rightarrow \text{quantise by replacing number-valued fields with operator-valued fields}$$

$$T^{\mu}_{\nu} \rightarrow \hat{T}^{\mu}_{\nu}$$

Then the classical momentum operator becomes an operator

$$P^i \rightarrow \hat{P}^i = - \int d^3\vec{x} \hat{\pi} \bar{\nabla}^i \hat{\phi} = \int \frac{d^3\vec{k}}{(2\pi)^3} k^i \hat{a}^{\dagger}(\vec{k}) \hat{a}(\vec{k})$$

↑ after normal ordering!

This operator has the correct properties

$$\bullet [\hat{P}^i, \hat{\phi}(\vec{x})] = i \bar{\nabla}^i \hat{\phi}(\vec{x})$$

$$\leftarrow [\hat{P}^i(y), \hat{\phi}(x)] = - \int d^3\vec{y} \frac{[\hat{\pi}(y), \hat{\phi}(x)] \partial^i \hat{\phi}(y)}{-i \delta^{ij}(x-y)}$$

• $\hat{a}^{\dagger}(\vec{k}) \hat{a}(\vec{k})$ is the "counting" or number operator

$$\hat{N} = \int \frac{d^3\vec{k}}{(2\pi)^3} \hat{a}^{\dagger}(\vec{k}) \hat{a}(\vec{k}) \quad \text{with} \quad \hat{N} |\bar{p}_1, \dots, \bar{p}_n\rangle = n |\bar{p}_1, \dots, \bar{p}_n\rangle$$

$$\text{and} \quad [\hat{H}, \hat{N}] = 0$$

↑ constructed via

$$|\bar{p}_1, \dots, \bar{p}_n\rangle = \hat{a}^{\dagger}(\bar{p}_1) \dots \hat{a}^{\dagger}(\bar{p}_n) |0\rangle$$

Only true for free theories!

↑ particle number is conserved

⇒ the momentum operator gives us the momentum \vec{k} times the number of particles with that momentum

• the momentum operator generates translations (in space)

$$\hat{T}(\vec{b}) = e^{-i \hat{P}_k b_k} \quad \text{such that} \quad \hat{T}(\vec{b}) |\vec{x}\rangle = |\vec{x} + \vec{b}\rangle$$

translation operator

$$\Rightarrow \hat{T}(\vec{b}) \hat{\phi}(\vec{x}) \hat{T}^{-1}(\vec{b}) = \hat{\phi}(\vec{x} + \vec{b})$$

$$\begin{aligned} \Rightarrow (1 - i \hat{P}_k \bar{b}_k) \hat{\phi}(\vec{x}) (1 + i \hat{P}_l \bar{b}_l) &= \hat{\phi}(\vec{x}) + b_i \bar{\nabla}_i \hat{\phi}(\vec{x}) \\ &\hookrightarrow = \hat{\phi}(\vec{x}) + b_i i (\hat{\phi}(\vec{x}) \hat{P}_i - \hat{P}_i \hat{\phi}(\vec{x})) \\ &= \hat{\phi}(\vec{x}) + b_i (-i [\hat{P}_i, \hat{\phi}(\vec{x})]) \end{aligned}$$

i.e. $[\hat{P}_i, \hat{\phi}(\vec{x})] = i \bar{\nabla}_i \hat{\phi}(\vec{x})$ is the requirement to generate translations

$$\bullet \hat{P} |\vec{k}\rangle = \hat{P} (\hat{a}^{\dagger}(\vec{k}) |0\rangle) = \vec{k} (\hat{a}^{\dagger}(\vec{k}) |0\rangle) = \vec{k} |\vec{k}\rangle$$

So far, so good. There is one untidy aspect we should clear up, though: normalisation.

We have defined $\langle 0|0\rangle = 1$ but what about $\langle k|p\rangle$?

$$\begin{aligned} \Rightarrow \langle 0|\hat{a}(\bar{k})\hat{a}^\dagger(\bar{p})|0\rangle &= \langle 0|[\hat{a}(\bar{k}), \hat{a}^\dagger(\bar{p})]|0\rangle \\ &= \delta^{(3)}(\bar{p}-\bar{k})\langle 0|0\rangle \\ &= \delta^{(3)}(\bar{p}-\bar{k}). \end{aligned}$$

↑ not Lorentz invariant $\ddot{\smile}$
(recall \sim volume)

To see this is not Lorentz invariant (and to lead us to the fix), consider a boost in the z -direction:

$$\begin{aligned} p_z' &= \gamma(p_z + \beta E) \Rightarrow \delta^{(3)}(\bar{p}-\bar{k}) = \delta^{(3)}(\bar{p}'-\bar{k}') \frac{dp_z'}{dp_z} \\ E' &= \gamma(E + \beta p_z) \\ &= \delta^{(3)}(\bar{p}'-\bar{k}') \gamma \left(1 + \beta \frac{dE}{dp_z}\right) \\ &= \delta^{(3)}(\bar{p}'-\bar{k}') \frac{\gamma}{E} (E + \beta p_z) \\ &= \delta^{(3)}(\bar{p}'-\bar{k}') \frac{E'}{E} \end{aligned}$$

\hookrightarrow tells us $E \delta^{(3)}(\bar{p}-\bar{k}) = E' \delta^{(3)}(\bar{p}'-\bar{k}')$ is invariant

\Rightarrow define $|\bar{k}\rangle = \sqrt{2E_k} \hat{a}^\dagger(\bar{k})|0\rangle$ so that $\langle \bar{k}|\bar{p}\rangle = 2E_k (2\pi)^3 \delta^{(3)}(\bar{p}-\bar{k})$

By choosing this normalisation we have to modify things elsewhere. For example, the identity or (projection operator onto) one-particle states is now

$$\hat{\mathbb{1}} = \int \frac{d^3\bar{k}}{(2\pi)^3} |\bar{k}\rangle \frac{1}{2E_k} \langle \bar{k}|$$

In fact $\int \frac{d^3\bar{k}}{(2\pi)^3} \frac{1}{2E_k}$ is a Lorentz invariant 3-momentum measure

because $\int \frac{d^3\bar{k}}{2k_0} \Big|_{k_0=E_k} = \int d^4k \delta^4(k_0^2 - \vec{k}^2 - m^2) \Big|_{k_0}$

P.T.O. for more complete proof

two solutions to $k^2 = m^2$
 $k_0^2 = E_k^2 = \vec{k}^2 + m^2$

Consider

manifestly covariant

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$\int d^4 p \theta(p^0) \delta(p^2 - m^2) f(p^\mu)$$

$$= \int d p^0 \theta(p^0) \int d^3 \bar{p} \delta((p^0)^2 - (\bar{p}^2 + m^2)) f(p^0, \bar{p})$$

$$= \int d p^0 \theta(p^0) \int d^3 \bar{p} \delta((p^0 - E_p)(p^0 + E_p)) f(p^0, \bar{p})$$

$$= \int d p^0 \theta(p^0) \int d^3 \bar{p} \delta(p^0 - E_p) \frac{1}{|p^0 + E_p|} f(p^0, \bar{p})$$

$$= \int d^3 \bar{p} \int_0^\infty d p^0 \frac{1}{p^0 + E_p} \delta(p^0 - E_p) f(p^0, \bar{p})$$

$$= \int d^3 p \frac{1}{2E_p} f(E_p, \bar{p}) \quad \#$$

We've been working in momentum space - what about coordinate space?

ie. what is $\hat{\phi}(\vec{x}, t) |0\rangle$?

We see that

$$\hat{\phi}(\vec{x}, t) |0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} e^{i\vec{k}\cdot\vec{x}} e^{-iE_k t} \hat{a}^\dagger(\vec{k}) |0\rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} e^{i\vec{k}\cdot\vec{x}} |\vec{k}\rangle$$

superposition of momentum eigenstates

Note this is not an eigenstate of position in our usual QM sense, because $\langle 0 | \hat{\phi}(\vec{y}, t') \hat{\phi}(\vec{x}, t) | 0 \rangle \neq \delta^{(3)}(\vec{x} - \vec{y})$

So we can project out a single momentum eigenstate

$$\langle \vec{p} | \hat{\phi}(\vec{x}, t) | 0 \rangle = \langle \vec{p} | \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} e^{i\vec{k}\cdot\vec{x}} |\vec{k}\rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} e^{i\vec{k}\cdot\vec{x}} 2E_k (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{k})$$

$$= e^{i\vec{p}\cdot\vec{x}} = e^{iE_p t - i\vec{p}\cdot\vec{x}}$$

the (complex conjugate of the) position space wavefunction for a free particle of momentum \vec{p} .

Fock spaces

So far we've largely been considering single particle states

We can construct arbitrary states via $\hat{a}^\dagger(\vec{p}_1) \hat{a}^\dagger(\vec{p}_2) \dots \hat{a}^\dagger(\vec{p}_n) |0\rangle$

The sum of all n -particle Hilbert spaces is a Fock space

Since all $\hat{a}^\dagger(\vec{p}_i)$ commute, all multiparticle states are symmetric

$$|p_1, p_2\rangle = |p_2, p_1\rangle$$

\Rightarrow our theory is a theory of bosons

| We can go further and show scalar fields are spin 0 - see Tong sec. 2.4 and 1.3.3 (15)

A summary

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - M^2 \phi^2) \Rightarrow (\partial^2 + M^2) \tilde{\phi}^2 = 0 \quad \text{Klein-Gordon eq.}$$

Real scalar field

$$\hat{\phi}(\vec{x}, t) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(e^{-ik_\mu x^\mu} \hat{a}(\vec{k}) + e^{ik_\mu x^\mu} \hat{a}^\dagger(\vec{k}) \right)$$

with conjugate momentum

$$\hat{\pi}(\vec{x}, t) = -i \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\sqrt{E_k}}{2} \left(e^{-ik_\mu x^\mu} \hat{a}(\vec{k}) - e^{ik_\mu x^\mu} \hat{a}^\dagger(\vec{k}) \right)$$

satisfy

$$[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)] = i \delta^{(3)}(\vec{x} - \vec{y}) \quad [\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{p})] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})$$

- Single particle states are $|\vec{k}\rangle = \hat{a}^\dagger(\vec{k}) |0\rangle$
and satisfy $\langle \vec{p} | \vec{k} \rangle = 2E_k (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{k})$

• Useful operators

- Hamiltonian
$$\hat{H} = \frac{1}{2} \int d^3 \vec{x} \left(\hat{\pi}^2 + \hat{\phi} (-\nabla^2 + M^2) \hat{\phi} \right)$$

$$\hat{H} = \int \frac{d^3 \vec{k}}{(2\pi)^3} E_k \hat{a}^\dagger(\vec{k}) \hat{a}(\vec{k})$$

satisfies $[\hat{H}, \hat{a}^{(\pm)}] = E_k \hat{a}^{(\pm)}$

$$\hat{H} |\vec{k}\rangle = E_k |\vec{k}\rangle$$

- Momentum operator
$$\hat{P}^i = - \int d^3 \vec{x} \hat{\pi} \nabla^i \hat{\phi}$$

$$\hat{P}^i = \int \frac{d^3 \vec{k}}{(2\pi)^3} k^i \hat{a}^\dagger(\vec{k}) \hat{a}(\vec{k})$$

satisfies $[\hat{P}^i, \hat{\phi}] = i \nabla^i \hat{\phi}$

$$\hat{P}^i |\vec{k}\rangle = \vec{k} |\vec{k}\rangle$$

- Counting operator
$$\hat{N} = \int \frac{d^3 \vec{k}}{(2\pi)^3} \hat{a}^\dagger(\vec{k}) \hat{a}(\vec{k})$$

satisfies $[\hat{H}, \hat{N}] = 0$

$$\hat{N} |p_1, \dots, p_n\rangle = n |p_1, \dots, p_n\rangle$$

Causality

free! = non-interacting

We've constructed a theory of quantum fields that have a relativistic dispersion relation (the Klein-Gordon equation).

But we haven't formulated everything in a Lorentz covariant way

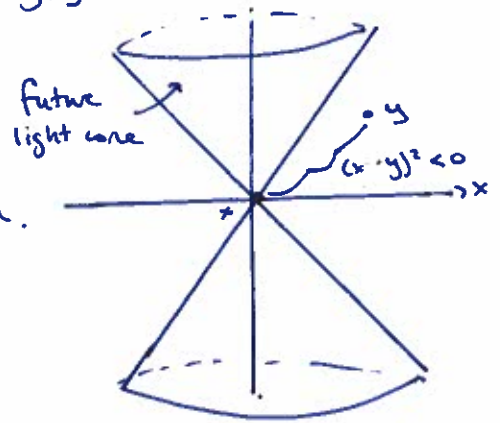
↑ recall equal-time commutation relations $[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y})$

We would like to formulate things for arbitrary times.

But our theory must respect causality

⇒ in QFT this means $[\hat{O}_1(x), \hat{O}_2(y)] = 0 \quad \forall (x-y)^2 < 0$

If this is not true then a measurement at x could affect a measurement at y , even though they are not causally connected.



Simplest example we can consider

$$\begin{aligned}
 [\hat{\phi}(x), \hat{\phi}(y)] &= \int \frac{d^3k d^3p}{(2\pi)^6} \frac{1}{2\sqrt{E_k E_p}} \left(e^{-ikx + ipy} [\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{p})] + e^{ikx - ipy} [\hat{a}^\dagger(\vec{k}), \hat{a}(\vec{p})] \right) \\
 &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left(e^{-ik \cdot (x-y)} - e^{ik \cdot (x-y)} \right) \\
 &\equiv \Delta(x-y) - \Delta(y-x)
 \end{aligned}$$

Here

$$\Delta(\vec{z}) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} e^{-ik \cdot z}$$

which we know is Lorentz covariant

- this object is a \mathbb{Q} -number
 - non-zero commutator for $(x-y)^2 \geq 0$
 - vanishing commutator for $(x-y)^2 < 0$
- ✓ true only for free theories

To see these last two properties

1. take $\bar{x} = 0, x^0 = 0$ and $\bar{y} = 0, y^0 - x^0 = t > 0$

↑ always possible to find Lorentz transformation to these coordinates from $y \in$ future lightcone

$$\begin{aligned} \Delta(0, t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{k^2+m^2}} e^{-it\sqrt{k^2+m^2}} \\ &= \frac{4\pi}{(2\pi)^3} \int_0^\infty dk \frac{k^2}{2\sqrt{k^2+m^2}} e^{-it\sqrt{k^2+m^2}} \\ &= \frac{1}{4\pi} \int_m^\infty dE \sqrt{E^2-m^2} e^{-iEt} \\ &\sim e^{-imt} \end{aligned}$$

$t \rightarrow \infty$

Then $[\phi(x), \phi(y)] \sim e^{-imt} - e^{imt} \neq 0 \quad (x-y)^2 \geq 0$

2. take $\bar{x} = 0, x^0 = 0$ and $\bar{x} - \bar{y} = \bar{r}, y^0 = 0$

↑ again always possible if $(x-y)^2 < 0$

$$\begin{aligned} \Delta(\bar{r}, 0) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{k^2+m^2}} e^{+ik \cdot \bar{r}} \\ &= \frac{1}{2} \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 \frac{1}{\sqrt{k^2+m^2}} \int_0^\pi d(\cos\theta) \int_0^{2\pi} d\phi e^{ikr \cos\theta} \quad |\bar{r} = |\bar{r}|| \\ &= \frac{1}{2(2\pi)^3} \int_0^\infty dk \frac{k^2}{\sqrt{k^2+m^2}} \frac{2\pi}{ikr} (e^{ikr} - e^{-ikr}) \\ &= \frac{-i}{8\pi^2 r} \int_0^\infty dk \frac{k}{\sqrt{k^2+m^2}} (e^{ikr} - e^{-ikr}) \quad \int k \rightarrow k' = -k \\ &= \frac{-i}{8\pi^2 r} \left(\int_0^\infty dk \frac{k e^{ikr}}{\sqrt{k^2+m^2}} - \int_0^{-\infty} dk \frac{k e^{ikr}}{\sqrt{k^2+m^2}} \right) \\ &= \frac{-i}{8\pi^2 r} \int_{-\infty}^\infty dk \frac{k e^{ikr}}{\sqrt{k^2+m^2}} \end{aligned}$$

$\sim e^{-mr} \neq 0$
 $r \rightarrow \infty$

see P+S pg 27-28

But $[\phi(x), \phi(y)] = \Delta(\bar{r}) - \Delta(-\bar{r}) \sim (e^{-mr} - e^{-mr}) = 0 \quad \# \quad (x-y)^2 < 0$

In general, the way that we probe a QFT (ie. make "measurements") in order to understand what is going on is by calculating correlation functions, which tell us about correlations (surprise surprise!) between different parts of our system. In other words, how does one part of our system affect another?

The simplest such correlation function we can consider (that isn't zero) is the two-point function

$$\langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle = \int \frac{d^3 \vec{p} d^3 \vec{q}}{(2\pi)^6} \frac{1}{\sqrt{4E_p E_q}} \langle 0 | \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{q}) | 0 \rangle e^{-ip \cdot x + iq \cdot y}$$

Now

$$\langle 0 | \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{q}) | 0 \rangle = \langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

↑ only nonvanishing term in expansion in terms of \hat{a}, \hat{a}^\dagger

N.B. $\langle \vec{p} | \vec{q} \rangle = 2E_p (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$
 ↓ 4-vectors!

← recall our discussion of the normalisation of states

so

$$\langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} = \Delta(x-y)$$

we saw this before!

This object is called a propagator

But as we saw it is not as useful as the combination $[\hat{\phi}(x), \hat{\phi}(y)]$ because it does not vanish outside the light cone.

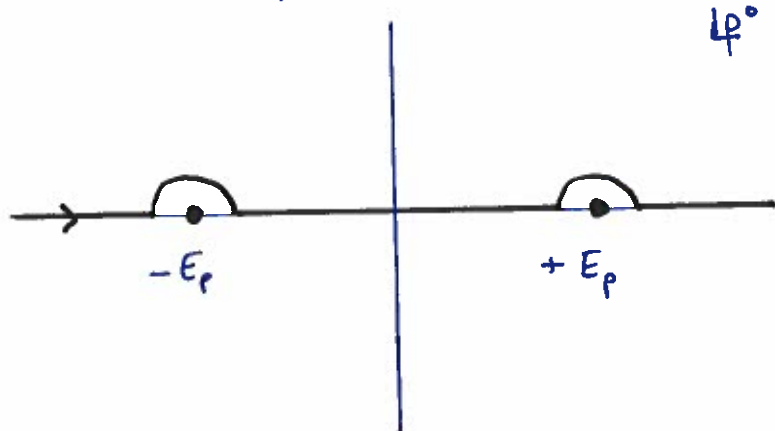
Propagator ~ probability amplitude for a particle to travel from x to y

Let's return to the combination that does satisfy causality

$$\begin{aligned} \langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle &= \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}) \\ &= \int \frac{d^3 \vec{p}}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip \cdot (x-y)} \Big|_{p^0=E_p} + \frac{1}{2(-E_p)} e^{-ip \cdot (x-y)} \Big|_{p^0=-E_p} \right\} \\ &= \int \frac{d^3 \vec{p}}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip \cdot (x-y)} \end{aligned}$$

$x^0 > y^0$

In the last line, we use the contour



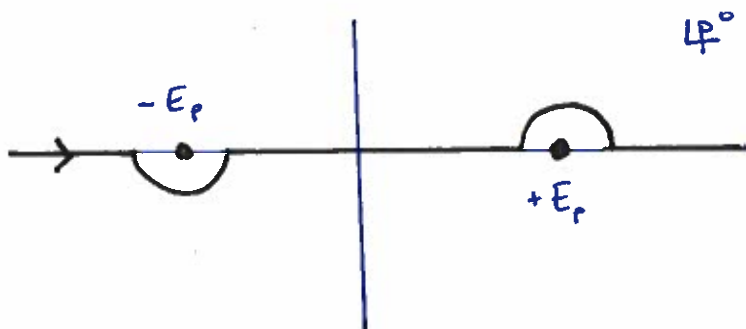
$x^0 > y^0$: close in upper half plane
 \Rightarrow non zero
 $x^0 < y^0$: close in lower half plane
 \Rightarrow zero

This object is another kind of propagator

$$D_{KG}(x-y) \equiv \theta(x^0 - y^0) \langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle$$

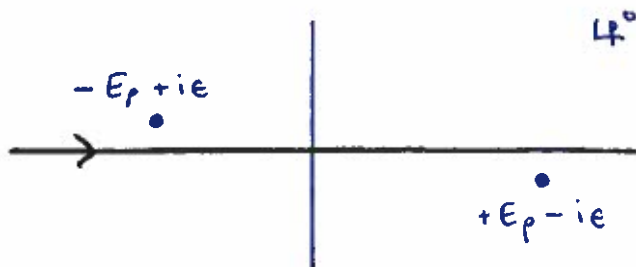
$$\theta(x^0 - y^0) = \begin{cases} 0 & x^0 < y^0 \\ 1 & x^0 > y^0 \end{cases}$$

But this is not the only choice of contour - there are four and each choice specifies a different function. The most useful will prove to be the Feynman propagator, defined by



which is entirely equivalent to

\uparrow i.e. move the poles not the contour!



Let's study this in more detail.

The Feynman propagator corresponds to the correlation function of the time-ordered product of fields, different from normal ordering!

Time ordering: place all fields in time-order, with the earliest to the right

$$T\{\hat{\phi}(\bar{x}, t_1) \hat{\phi}(\bar{y}, t_2)\} \equiv \theta(t_1 - t_2) \hat{\phi}(\bar{x}, t_1) \hat{\phi}(\bar{y}, t_2) + \theta(t_2 - t_1) \hat{\phi}(\bar{y}, t_2) \hat{\phi}(\bar{x}, t_1)$$

In fact

$$\begin{aligned} D_F(x-y) &= \langle 0 | T\{\phi(x) \phi(y)\} | 0 \rangle \\ &= \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle \\ &= \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} \end{aligned}$$

To show the last step, let's consider the pole structure

$$\begin{aligned} p^2 - m^2 + i\epsilon &= (p^0)^2 - \vec{p}^2 - m^2 + i\epsilon && \overline{|\epsilon \ll 1|} \\ &= (p^0)^2 - E_p^2 + i\epsilon \\ &= (p^0 - \sqrt{E_p^2 - i\epsilon})(p^0 + \sqrt{E_p^2 - i\epsilon}) \\ &\approx \underbrace{(p^0 - (E_p - i\epsilon) + O(\epsilon^2))}_{\text{poles as in diagram on previous page}} \underbrace{(p^0 + (E_p - i\epsilon) + O(\epsilon^2))} \end{aligned}$$

Now let's take $x^0 > y^0$

$$\Rightarrow e^{-ip \cdot (x-y)^0} = e^{-i \operatorname{Re}(p^0) |x^0 - y^0|} e^{+ \operatorname{Im}(p^0) |x^0 - y^0|}$$

\Rightarrow we close the contour in the lower half plane (where $\operatorname{Im}(p^0) < 0$) and the contour at infinity contributes zero

So our integral becomes

$$\begin{aligned} \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} &= i \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{+i\vec{p} \cdot (\vec{x} - \vec{y})} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0 - y^0)}}{(p^0 - (E_p - i\epsilon))(p^0 + (E_p - i\epsilon))} \\ &= i \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{+i\vec{p} \cdot (\vec{x} - \vec{y})} (-2\pi i) \text{Residue}_{p^0 = E_p - i\epsilon} \left\{ \int \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0 - y^0)}}{(p^0 - (E_p - i\epsilon))(p^0 + (E_p - i\epsilon))} \right\} \\ &= i \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{+i\vec{p} \cdot (\vec{x} - \vec{y})} \left(\frac{-2\pi i}{2\pi} \right) \frac{e^{-i(E_p - i\epsilon)(x^0 - y^0)}}{2(E_p - i\epsilon)} \quad \begin{array}{l} \uparrow \\ \text{clockwise} \\ \text{contour} \Rightarrow -2\pi i! \end{array} \\ \stackrel{\epsilon \rightarrow 0}{=} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)} &= \Delta(x-y) \end{aligned}$$

Next we take $y^0 > x^0$

$$\Rightarrow e^{-ip \cdot (x-y)} = e^{ip \cdot |y-x|} = e^{i\text{Re}(p^0)|y^0 - x^0|} e^{-\text{Im}(p^0)|y^0 - x^0|}$$

\Rightarrow we close the contour in the upper half plane (where $\text{Im}(p^0) > 0$) and again the contour at infinity gives zero

In this case our integral becomes

$$\begin{aligned} \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} &= i \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{+i\vec{p} \cdot (\vec{x} - \vec{y})} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0 - y^0)}}{(p^0 - (E_p - i\epsilon))(p^0 + (E_p - i\epsilon))} \\ &= i \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{+i\vec{p} \cdot (\vec{x} - \vec{y})} (2\pi i) \text{Residue}_{p^0 = -E_p + i\epsilon} \left\{ \int \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0 - y^0)}}{(p^0 - (E_p - i\epsilon))(p^0 + (E_p - i\epsilon))} \right\} \\ &= i \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{+i\vec{p} \cdot (\vec{x} - \vec{y})} \frac{2\pi i}{2\pi} \frac{e^{-i(-E_p + i\epsilon)(x^0 - y^0)}}{-2(E_p - i\epsilon)} \\ &= \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{+i\vec{p} \cdot (\vec{x} - \vec{y})} e^{ip^0(x^0 - y^0)} = \Delta(y-x) \\ &\quad \downarrow \\ &\quad e^{ip^0(x^0 - y^0)} = e^{-ip^0(y^0 - x^0)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} &= \theta(x-y) \Delta(x-y) + \theta(y-x) \Delta(y-x) \\ &= D_F(x-y) \end{aligned}$$

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There is another interpretation of the Feynman propagator that illustrates its utility. Note:

$$(\partial_x^2 + M^2) D_F(x-y) = (\partial_x^2 + M^2) i \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip \cdot (x-y)}}{p^2 - M^2 + i\epsilon}$$

$$= i \int \frac{d^4 p}{(2\pi)^4} \frac{(-p^2 + M^2)}{p^2 - M^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

$$= -i \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)}$$

$$= -i \delta^{(4)}(x-y)$$

↖ $\epsilon \rightarrow 0$ limit
will be always
understood
in future

⇒ Feynman propagator is a Green's function
of the Klein-Gordon operator

See P+S pg. 30
for proof in time-ordered
form (careful with the
 θ -function!).

So are D_{KG} and Δ ,
but D_F is the
most useful

This is all great, but so far we have only
considered free fields - free fields essentially
go about their business, propagated around the
plane, as they do. All the interesting stuff happens
when we incorporate interactions.

A summary

- Propagators are Green's functions of the Klein-Gordon operator.

$$O_{KG} \cdot D(x-y) \propto \delta^{(4)}(x-y)$$

- Two-point function

$$\langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)} \equiv \Delta(x-y)$$

- Klein-Gordon propagator

$$\langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle = \Delta(x-y) - \Delta(y-x)$$

$$D_{KG}(x-y) \equiv \theta(x^0 - y^0) \langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle$$

$$=_{x^0 - y^0 > 0} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip \cdot (x-y)}$$

- Feynman propagator

$$\langle 0 | T \{ \hat{\phi}(x), \hat{\phi}(y) \} | 0 \rangle = \theta(x-y) \Delta(x-y) - \theta(y-x) \Delta(x-y)$$

$$D_F(x-y) = \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

