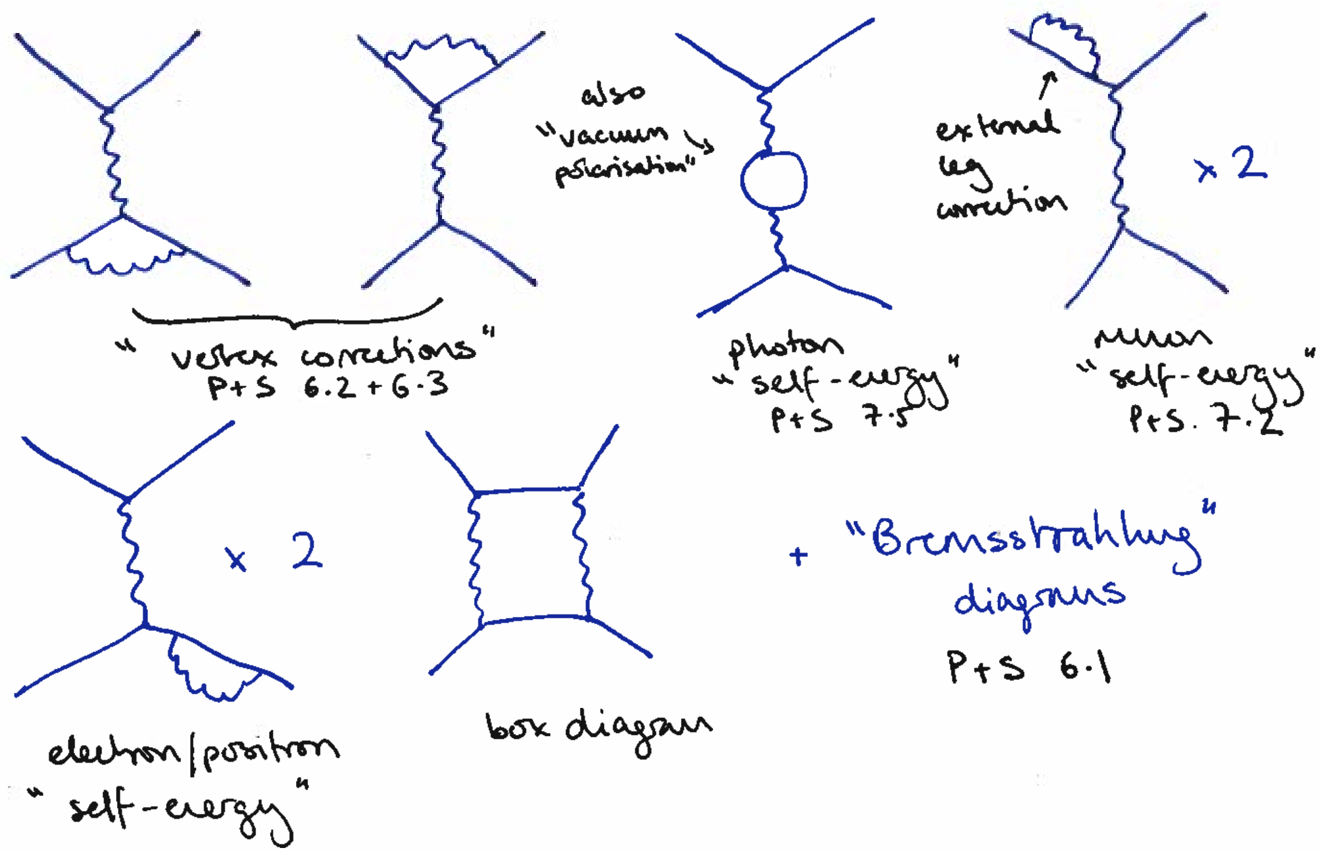
Radiative Corrections

Our discussion of scattering has been focussed on the leading order, or "tree-level", diagrams. But this is, literally, just the start of quantum field theory. To start to harness the full power of quantum field theory, we need to include quantum corrections or "radiative corrections".

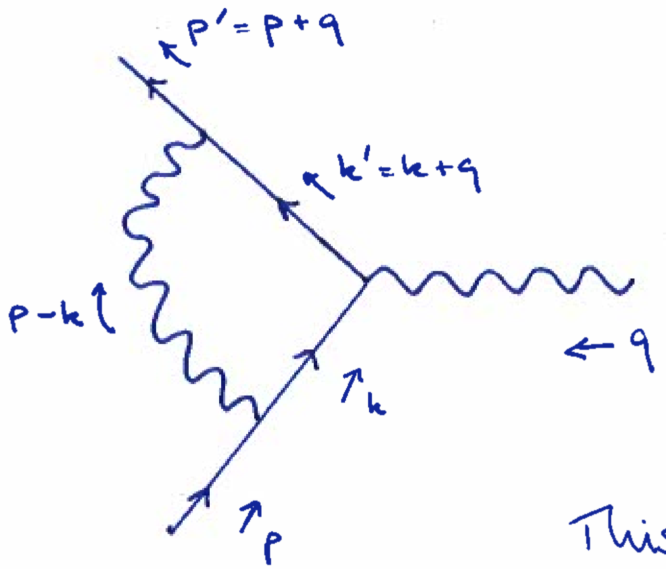
We will start our discussion with the next-to-leading order corrections in our favourite process: $e^+e^- \to \mu^+\mu^-$



+ "Bremsstrahlung" diagrams
P+S 6.1

We will focus on the vertex corrections, and, specifically, the electron vertex correction, as that is dealt with by P+S in 6.2 and 6.3.

Note tree-level contribution is
$\Gamma^\mu_{(0)} = \bar{u}(-ig\gamma^\mu)u$



Note that we don't assume that $q^2 = 0$.
In other words, the photon can be "off-shell".

It's also not external!

This diagram has both
- infrared divergences
- ultraviolet divergences

$$\Gamma^\mu_{(1)} = \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\nu\rho}}{(p-k)^2 + i\epsilon} \bar{u}(p')(-ig\gamma^\rho) \frac{i(\not{k}'+m)}{k'^2 - m^2 + i\epsilon} (-ig\gamma^\mu)$$
$$\times \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ig\gamma^\nu)u(p)$$

$$= (-ig)^3 i(-i)\bar{u}(\bar{p}') \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\rho(\not{k}'\gamma^\mu\not{k} + m\gamma^\mu\not{k} + \not{k}'\gamma^\mu m + m^2\gamma^\mu)\gamma_\rho u(\bar{p})}{((p-k)^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}$$

Using the contraction indices, this becomes
$$= ig^3\bar{u}(\bar{p}') \int \frac{d^4k}{(2\pi)^4} \frac{(-2)(\not{k}\gamma^\mu\not{k}' - 2m(k+k')^\mu + m^2\gamma^\mu)u(\bar{p})}{((p-k)^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}$$

To solve this integral, which, frankly looks a mess, we need the technique of "Feynman parameters". But first let's motivate these techniques by looking at a simpler integral.

$$I_1 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k-p)^2 - m^2 + i\epsilon} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+p^2-2k\cdot p - m^2 + i\epsilon}$$

↑ makes our life hard! [angular dependence]

Solve by defining $\ell^\mu = k^\mu - p^\mu$

$\Rightarrow I_1 = \int \frac{d^4\ell}{\ldots} \frac{1}{\ldots}$ ← no angular dependence!

What about the slightly more complicated integral

$$I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k-p)^2} \frac{1}{(k^2-m^2)}$$

$$= \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2} \frac{1}{(\ell+p)^2-m^2}$$

This doesn't work - we still have some angular dependence. We need to find a way to "complete the square" to eliminate any angular dependence. And the way to do this is through feynman parameters:

Example

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \; \delta(x+y-1) \frac{1}{[xA+yB]^2}$$

$$= \int_0^1 dx \frac{1}{[xA+(1-x)B]^2}$$

$\Rightarrow$ in our case

$$I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k-p)^2} \frac{1}{(k^2-m^2)}$$

$$= \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{[x(k-p)^2+(1-x)(k^2-m^2)]}$$

$$= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{D(k)}$$

Now $D(k) = x(k^2-2p\cdot k+p^2) + (1-x)(k^2-m^2)$

$\qquad\quad = k^2 - x2p\cdot k + xp^2 - (1-x)m^2$

P+S generally use Δ not Δ^2, but this way $[\Delta]=[m]$.

Let $\ell^\mu = k^\mu - xp^\mu \Rightarrow \ell^2 = k^2 - 2xp\cdot k + x^2p^2$

$\Rightarrow D(k) = \ell^2 - x^2p^2 + xp^2 - (1-x)m^2 \equiv \ell^2 - \Delta^2(x) \equiv D(\ell)$

So we can write

$$I_2 = \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta^2)^2}$$ no angular dependence!

For more than two propagators, we have

$$\boxed{\frac{1}{A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n}} = \int_0^1 dx_1 \ldots dx_n \, \delta\left(\sum_i x_i - 1\right) \frac{\prod_i x_i^{m_i - 1}}{\left[\sum_i x_i A_i\right]^{\sum m_i}} \frac{\Gamma(m_1 + \ldots + m_n)}{\Gamma(m_1) \ldots \Gamma(m_n)}}$$

↑ m_i need not be integers

P+S sketch proof on p. 190.

Returning to the vertex function, the expression we need is

$$\frac{1}{((k-p)^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{2}{D^3}$$

Here

$$\begin{aligned}
D &= x(k^2 - m^2 + i\epsilon) + y((k+q)^2 - m^2 + i\epsilon) + z((p-k)^2 + i\epsilon) \\
&= i\epsilon(x+y+z) + k^2(x+y+z) + 2k\cdot q y - 2k\cdot p z + q^2 y + p^2 z - m^2(x+y) \\
&= k^2 + 2k\cdot(yq - pz) + q^2 y + p^2 z - m^2(x+y) + i\epsilon \qquad x+y+z=1
\end{aligned}$$

Now let $l^\mu = k^\mu + yq^\mu - zp^\mu$, so $l^2 = k^2 + 2k\cdot(yq - pz) + (yq - zp)^2$

$$\begin{aligned}
\Rightarrow D &= l^2 - (yq - zp)^2 + q^2 y + p^2 z - m^2(1-z) + i\epsilon \\
&= l^2 + q^2(y - y^2) + p^2(z - z^2) + 2p\cdot q\, yz - m^2(1-z) + i\epsilon \qquad p^2 = m^2 \\
&= l^2 + q^2 y(1-y) + m^2 z(1-z) - m^2(1-z) + 2p\cdot q\, yz + i\epsilon \\
&= l^2 + q^2 y(1-y) - m^2(z-1)^2 + 2p\cdot q\, yz + i\epsilon
\end{aligned}$$

Now we can use momentum conservation, so

$$\begin{aligned}
p' = p + q \Rightarrow p'^2 &= p^2 + q^2 + 2p\cdot q \\
\Rightarrow m^2 &= m^2 + q^2 + 2p\cdot q \\
\Rightarrow 2p\cdot q &= -q^2
\end{aligned}$$

Plugging this in, we get

$$D = \ell^2 + q^2 y(1-y) - m^2(z-1)^2 - q^2 yz + i\epsilon$$
$$= \ell^2 + q^2 y(1-y-z) - m^2(z-1)^2 + i\epsilon$$
$$= \ell^2 + q^2 yx - m^2(z-1)^2 + i\epsilon \qquad \& \ \Delta^2 = -q^2 yx + m^2(1-z)^2$$
$$\equiv \ell^2 - \Delta^2 + i\epsilon$$

P+S eq. 6.44

That's the denominator sorted, what about the numerator

$$\Gamma^\mu_{(num)} = \bar{u}(\bar{p}')\left(\not{k}\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right)u(\bar{p})$$
$$= \bar{u}(\bar{p}')\left(\not{k}\gamma^\mu(\not{k}+\not{q}) + m^2\gamma^\mu - 2m(2k+q)^\mu\right)u(\bar{p})$$

Now let $\ell^\mu = k^\mu + yq^\mu - zp^\mu$

$$\Gamma^\mu_{(num)} = \bar{u}(\bar{p}')\Big[(\not{\ell} - (y\not{q} - z\not{p}))\gamma^\mu(\not{\ell} - (y-1)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m(2\ell + (1-2y)q + 2zp)^\mu\Big]u(\bar{p})$$

$$= \bar{u}(\bar{p}')\Big[\not{\ell}\gamma^\mu\not{\ell} - (y\not{q} - z\not{p})\gamma^\mu\not{\ell} - \not{\ell}\gamma^\mu((y-1)\not{q} - z\not{p}) - 4m\ell^\mu - (y\not{q} - z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu)\Big]u(\bar{p})$$

$$\equiv \bar{u}(\bar{p}')\left[A^{(2)} + A^{(1)} + A^{(0)}\right]u(\bar{p})$$

superscript denotes number of powers of ℓ^μ.

So our expression is

$$\Gamma^\mu_{(1)} = 4ig^3\,\bar{u}(\bar{p}')\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\frac{\left[A^{(2)} + A^{(1)} + A^{(0)}\right]u(\bar{p})}{(\ell^2 - \Delta^2 + i\epsilon)^3}$$

Our task now is to calculate the loop integrals $J_3^{(i)}$.

The easier one is

$$J_3^{(0)} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta^2 + i\epsilon)^3}$$

Two ways to do this:

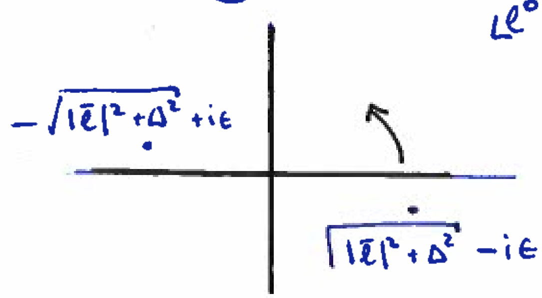
1. Contour integration in l_0 and then the spatial integral (hard).
2. Hyperspherical (4D) coordinates (much easier)

We will use the second method, but first we need to "Wick rotate" - transform the l^0 coordinate so that $l^2 = l^{0\,2} + l^{x\,2} + l^{y\,2} + l^{z\,2}$. We achieve this by defining

$$l^0 \equiv i l_E^0 \qquad l^i = l_E^i \quad \leftarrow \text{"Euclidean momentum"}$$

$$\Rightarrow d^4 l = i\, d^4 l_E \qquad \uparrow \text{ Euclidean metric } g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is allowed because we do not cross any poles in the l^0 plane when we Wick rotate



Then
$$\begin{aligned} l^2 - \Delta^2 &= l^{0\,2} - |\bar{l}|^2 - \Delta^2 \\ &\to (i l_E^0)^2 - |\bar{l}_E|^2 - \Delta^2 \\ &= -(l_E^0)^2 - |\bar{l}_E|^2 - \Delta^2 \\ &= -(l_E^2 + \Delta^2) \end{aligned}$$

So
$$J_3^{(0)} = \frac{i}{(-1)^3} \frac{1}{(2\pi)^4} \int d^4 l_E \frac{1}{(l_E^2 + \Delta^2)^3} = \frac{-i}{(2\pi)^4} \int d\Omega_4 \int_0^\infty dl_E \frac{l_E^3}{(l_E^2 + \Delta^2)^3}$$

The denominator is even under $\ell^\mu \to -\ell^\mu$, so we can immediately drop the term $A^{(1)}$, as this is odd under $\ell^\mu \to -\ell^\mu$. The other two terms are even, however, so let's look at those.

First we note that the $A^{(2)}$ term involves the integral

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\not\ell \gamma^\mu \not\ell}{(\ell^2-\Delta^2+i\epsilon)^3} = \int \frac{d^4\ell}{(2\pi)^4} \frac{2\ell^\mu \not\ell - \ell^2\gamma^\mu}{(\ell^2-\Delta^2+i\epsilon)}$$

$$= 2\gamma_\alpha \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\alpha}{(\ell^2-\Delta^2+i\epsilon)^3} - \gamma^\mu \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta^2+i\epsilon)^3}$$

$$\begin{aligned} \not a \gamma^\mu \not a &= a_\alpha a_\beta \gamma^\alpha \gamma^\mu \gamma^\beta \\ &= a_\alpha a_\beta (2g^{\alpha\mu}\gamma^\beta - \gamma^\mu \gamma^\alpha \gamma^\beta) \\ &= 2a^\mu \not a - \gamma^\mu \not a \not a \\ &= 2a^\mu \not a - a^2 \gamma^\mu \end{aligned}$$

$$= 2\gamma_\alpha \frac{1}{4} g^{\mu\alpha} \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta^2+i\epsilon)^3} - \gamma^\mu \int \frac{d^4\ell}{(2\pi)^3} \frac{\ell^2}{(\ell^2-\Delta^2+i\epsilon)^3}$$

$$= -\frac{1}{2}\gamma^\mu \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta^2+i\epsilon)^3}$$

integral $= 0$ if $\mu \neq \nu$

$$\int_\ell \frac{\ell^\mu \ell^\nu}{D^3} = C g^{\mu\nu} \int_\ell \frac{\ell^2}{D^3}$$

$$\Rightarrow g_{\mu\nu} \int_\ell \frac{\ell^\mu \ell^\nu}{D^3} = C g_\mu{}^\mu \int_\ell \frac{\ell^2}{D^3}$$

$$\Rightarrow \int_\ell \frac{\ell^2}{D^3} = C \cdot 4 \int_\ell \frac{\ell^2}{D^3}$$

$$\Rightarrow 4C = 1$$

Now the $A^{(0)}$ term has no ℓ-dependence in the numerator. Therefore the integral for this term is just

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta^2+i\epsilon)^3}.$$

We can now write our expression as

$$\Gamma^\mu_{(1)} = 4ig^3 \bar{u}(\bar{p}') \int_0^1 dx\,dy\,dz\, \delta(x+y+z-1) \left[-\frac{1}{2}\gamma^\mu J_3^{(2)} + A^{(0)} J_3^{(0)}\right] u(\bar{p})$$

where

$$J_3^{(2)} = \int \frac{d^4\ell}{(2\pi)^3} \frac{\ell^2}{(\ell^2-\Delta^2+i\epsilon)^3} \qquad J_3^{(0)} = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta^2+i\epsilon)^3}$$

$$A^{(0)} = -(y\not q - z\not p)\gamma^\mu((1-y)\not q + z\not p) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu)$$

Here $\int d\Omega_4 = \int \sin^2\omega \sin\theta \, d\phi \, d\theta \, d\omega$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\pi} \sin^2\omega \, d\omega$$

$$= 2\pi^2$$

"surface area" of a 4D sphere

N.B measure comes from 4D hyperspherical coordinates

$$x = r(\sin\omega \sin\theta \cos\phi, \sin\omega \sin\theta \sin\phi, \sin\omega \cos\theta, \cos\omega)$$

$$\Rightarrow d^4x = r^3 \sin^2\omega \sin\theta \, d\phi \, d\theta \, d\omega \, dr$$

More generally, in d-dimensions

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

d	$\Gamma(d/2)$	$\int d\Omega_d$
1	$\sqrt{\pi}$	2
2	1	2π
3	$\sqrt{\pi}/2$	4π
4	1	$2\pi^2$

P+S p. 249

This can be proved by considering

$$\left(\int dx \, e^{-x^2}\right)^d = (\sqrt{\pi})^d$$

$$= \int d^dx \, e^{-\sum_{i=1}^{d} x_i^2}$$

$$= \int d\Omega_d \int_0^{\infty} dx \, x^{d-1} e^{-x^2}$$

$$= \int d\Omega_d \cdot \frac{1}{2}\Gamma(d/2)$$

$$\Rightarrow \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

Note $\Delta^2 > 0$

Thus our integral is

$$J_3^{(0)} = \frac{-i}{(2\pi)^4} 2\pi^2 \int_0^{\infty} d\ell_E \frac{\ell_E^3}{(\ell_E^2 + \Delta^2)^3} = -\frac{i}{8\pi^2} \cdot \frac{1}{4\Delta^2} = \frac{-i}{32\pi^2\Delta^2}$$

More generally,

$$J_m^{(0)} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta^2 + i\epsilon)^m}$$

$$= \frac{i(-1)^m}{(4\pi)^2} \frac{1}{(m-1)(m-2)} \frac{1}{(\Delta^2)^{m-2}}$$

What about our other integral?

$$J_3^{(2)} = \frac{i}{(-1)^3} \frac{1}{(2\pi)^4} \int d\Omega_4 \int_0^\infty dl_E \frac{l_E^2 \cdot l_E^3}{(l_E^2 + \Delta^2)^3}$$

$$= -\frac{i}{8\pi^2} \int_0^\infty dl_E \frac{l_E^5}{(l_E^2 + \Delta^2)^3}$$

But this is divergent! If we consider the slightly more general case

$$J_m^{(2)} = \int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta^2 + i\epsilon)^m}$$

$$= \frac{i(-1)^{m-1}}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)} \frac{1}{(\Delta^2)^{m-3}}$$

clearly requires $m > 3$ ☹

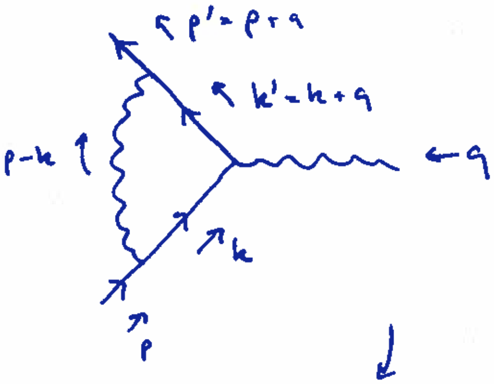
In particular, this integral is "<u>ultraviolet</u>" (UV) divergent – the divergence comes from large momenta values (or very small distances). Before we discuss how to fix this divergence, let's pause to discuss different types of divergence that we might encounter:

1. Ultraviolet (UV)
   - arise from quantum effects (short distances / large momenta)
   - these are "unphysical"
   - made finite by "<u>regularisation</u>"
   - absorbed into "<u>renormalisation</u>" procedure

Important to distinguish the regulator from renormalisation!!

Recap

Studying the radiative correction to $\Gamma^\mu_{(0)} = g\bar{u}(\bar{p}')\gamma^\mu u(\bar{p})$



$\longrightarrow \Gamma^\mu_{(1)}$

We used:

- Feynman parameters
- Wick rotation
- Dimensional regularisation

to write this as

$$\Gamma^\mu = 4ig^3\,\bar{u}(\bar{p}')\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[-\tfrac{1}{2}\gamma^\mu J_3^{(2)} + A^{(0)} J_3^{(0)}\right]u(\bar{p})$$

where

$$J_3^{(0)} = \int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta^2+i\epsilon)^3} = \frac{-i}{(4\pi)^2 2\Delta^2}$$

$$J_3^{(2)} = \int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta^2+i\epsilon)^3} \quad \leftarrow \underline{\text{UV divergent}}$$

and

$$A^{(0)} = -(y\not{q} - z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu)$$

2. Infrared (IR) – these are "physical" ← long distances, low momenta
– not absorbed into renormalisation procedure
– rendered finite by physical mechanism
  • angle of resolution in detector
  • confinement in QCD

N.B. We first saw these terms in the context of normal ordering. P. 11 of notes.

In the remainder of this course, we'll focus on UV divergences.

To deal with these, we first need to render the integral finite, using a regulator:

Also: • analytic regularisation – $\frac{1}{(p^2+m^2)^a}$

- cutoff – We could simply integrate up to some cutoff, rather than infinity. Obviously the result will then depend on the cutoff (we deal with this via renormalisation). This has a major problem – it is not gauge invariant.

  simplest

- Pauli-Villars – Peskin and Schroeder's choice in this chapter. We introduce a fictitious heavy particle that effectively cuts off momenta larger than the fictitious heavy mass

- Dimensional regularisation – Gauge invariant, and straightforward to implement (in the absence of γ^5). Does obscure distinction between IR and UV divergences, and only defined within perturbation theory.

  most widespread

[• Lattice QFT – Gauge invariant and nonperturbative. Significantly complicates perturbation theory.]

Let's now apply the first three of these regulators to our integral $J_3^{(2)}$

$J_m^{(2)}$ can be done analytically, but the results are not elegant

Recall

$$J_3^{(2)} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta^2+i\epsilon)^3}$$

- Cutoff

$$J_3^{(2)} = \frac{i}{(-1)} \int \frac{d\Omega_4}{(2\pi)^4} \int_0^\Lambda d\ell_E \frac{\ell_E^5}{(\ell_E^2+\Delta^2)^3}$$

$$= (-i) \frac{2\pi^2}{(2\pi)^4} \left[-\frac{3\Lambda^4+2\Lambda^2\Delta^2}{4(\Lambda^2+\Delta^2)^2} + \frac{1}{2}\log\left(1+\frac{\Lambda^2}{\Delta^2}\right)\right]$$

$$= \frac{-i}{(4\pi)^2}\left[\log\left(\frac{1+\delta}{\delta}\right) - \frac{3}{2}\frac{1+2\delta^2}{(1+\delta^2)^2}\right] \qquad \delta \equiv \frac{\Delta^2}{\Lambda^2}$$

In the limit $\delta \ll 1$ (ie the cutoff is much larger than all other scales), we have

$$J_3^{(2)} \xrightarrow[\delta \ll 1]{} \frac{i}{(4\pi)^2}\left[\frac{3}{2} + \log\frac{\Delta^2}{\Lambda^2}\right]$$

- logarithmic divergence typical of UV divergences
- divergence type is dictated by mass dimension of integrand
  $\int d^4\ell \frac{\ell^2}{(\ell^2)^3} \sim \int d\ell \frac{\ell^5}{\ell^6} \sim \int \frac{d\ell}{\ell} \sim \log\Lambda$
- "power divergences" are possible, but beyond the scope of this course
  $\int d^4\ell \frac{(\ell^2)^2}{(\ell^2)^3} \sim \int d\ell \frac{\ell^7}{\ell^6} \sim \int \ell\, d\ell \sim \Lambda^2$

- Pauli-Villars

introduce "fictitious heavy photon", whose effect is subtracted from the photon

Replace propagator with

$$\frac{1}{(k-p)^2+i\epsilon} \longrightarrow \frac{1}{(k-p)^2+i\epsilon} - \frac{1}{(k-p)^2-\Lambda^2+i\epsilon}$$

Then it turns out

$$\Delta^2 \to \Delta_\Lambda^2 = -xyq^2 + (1-z)^2 m^2 + z\Lambda^2$$

and we have

$$J_3^{(2)} = \int \frac{d^4 l}{(2\pi)^4}\, l^2 \left[\frac{1}{[l^2-\Delta^2]^3} - \frac{1}{[l^2-\Delta_\Lambda^2]^3}\right]$$

$$= \frac{i}{(4\pi)^2} \int_0^\infty d^2 l_E \left(\frac{l_E^4}{(l_E^2+\Delta^2)^3} - \frac{l_E^4}{(l_E^2+\Delta_\Lambda^2)^3}\right)$$

$$= \frac{i}{(4\pi)^2} \log\left(\frac{\Delta_\Lambda^2}{\Delta^2}\right)$$

- Dimensional regularisation

In this case we replace our four dimensional integral with a d-dimensional integral, where we usually take $d = 4-2\epsilon$ $(\epsilon > 0)$. Divergences now appear as <u>poles in ϵ</u>.

N.B. For IR divergences we need $\epsilon < 0$.

Thus our integral becomes

$$J_3^{(2)} = \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta^2+i\epsilon)^3}$$

$[\mu] = M \leftarrow$ this is added to fix up the dimensions of the integral

Then

$$J_3^{(2)} = \frac{i}{(-1)^3} \frac{\mu^{4-d}}{(2\pi)^d} \int d\Omega_d \int d^{d-1}\ell_E \frac{\ell_E^2}{(\ell_E^2+\Delta^2)^3}$$

$$= \frac{-i}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \mu^{4-d} \int_0^\infty d\ell_E \frac{\ell_E^{d-1}\cdot \ell_E^2}{(\ell_E^2+\Delta^2)^3}$$

$y = \ell^2 \Rightarrow \frac{1}{2}dy = d\ell \cdot \ell$

$$= \frac{-i}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \mu^{4-d} \frac{1}{2} \int_0^\infty d\ell_E^2 \frac{(\ell_E^2)^{d/2}}{(\ell_E^2+\Delta^2)^3}$$

$$= \frac{-i}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \mu^{4-d} \frac{1}{2} (\Delta^2)^{d/2-2} \int_0^1 dx\, x^{1-d/2}(1-x)^{d/2}$$

N.B. $x = \frac{\Delta^2}{\ell^2+\Delta^2}$

$\Rightarrow dx = -\frac{\Delta^2 d\ell^2}{(\ell^2+\Delta^2)^2}$

$\ell^2 = \Delta^2 x^{-1}(1-x)$

And minus sign cancelled by switching integration limits.

This form is useful because

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

In our case, we have

$$J_3^{(2)} = \frac{-i\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \mu^{4-d} (\Delta^2)^{d/2-2} \frac{\Gamma(2-d/2)\Gamma(1+d/2)}{\Gamma(3)}$$

$\frac{\Gamma(1+d/2)}{\Gamma(d/2)} = \frac{d}{2}$

$\Gamma(3) = 2$

$$= \frac{-i}{4(4\pi)^{d/2}} \mu^{4-d} (\Delta^2)^{d/2-2} \frac{d}{2} \Gamma(2-d/2)$$

$\Gamma(z)$ has poles at $z = 0, -1, -2, \ldots$

$\Rightarrow$ poles at $d = 4, 6, 8, \ldots$

We are interested in the case $d=4$, but this diverges, so let's look in the neighbourhood, by choosing $d = 4-\epsilon$

N.B. Here $\epsilon > 0$, so for IR divergences we can choose $d = 4+\epsilon$.

Now

Use $\dfrac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty}\left(1+\dfrac{z}{n}\right)e^{-z/n}$

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right)$$
$$= \frac{2}{\epsilon} - \gamma_E + O(\epsilon)$$

$\gamma_E \simeq 0.5772$ is the Euler-Mascheroni constant

Now we substitute $d = 4-\epsilon$ into our expression and expand

$$J_3^{(2)} = \frac{-i}{(4\pi)^2}\frac{1}{4}\left(\frac{2}{\epsilon} - \gamma_E - \log\frac{\Delta^2}{\mu^2} + \log 4\pi\right)$$

pole in ϵ corresponds to logarithmic divergence in Pauli-Villars and cutoff regularisations.

Some comments on dimensional regularisation

- scale-free integrals vanish

  e.g. $\displaystyle\int \frac{d^dk}{k^n} = 0$

  N.B. No power divergences in dim reg!

  to see this, split integral

  $$\int \frac{d^dk}{k^n} \cdot \int_0^{\Lambda} dk\, k^{d-1-n} + \int_{\Lambda}^{\infty} dk\, k^{d-1-n}$$
  $$= \underbrace{\frac{\Lambda^{d-n}}{d-n}}_{d>n} - \underbrace{\frac{\Lambda^{d-n}}{d-n}}_{d<n} = 0$$

- we must generalise our gamma matrices to d-dimensions

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad \mathrm{Tr}(\mathbb{1}) = 4 \qquad g^{\mu\nu}g_{\mu\nu} = d$$

See P+S Appendix A.4 for useful summary

$$\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma$$

- if numerator contains $\ell^\mu\ell^\nu$ we replace it by $\frac{1}{d}\ell^2 g^{\mu\nu}$

  and $\ell^\mu\ell^\nu\ell^\rho\ell^\sigma$ by $\dfrac{1}{d(d+2)}(\ell^2)^2\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$

- Warning: γ^5 cannot be treated consistently in d-dimensions! This is intimately connected to anomalies and chiral symmetry, but far beyond this course.

See, e.g. hep-th/0005255
or Nucl Phys. B. 333 (1990) 66
or Commun. Math. Phys 52 (1977) 11
or ...