Quantum Field Theory I: Physics 721

Autumn/Fall 2020

Why QFT?

Three basic reasons

1. Preserves locality $\begin{cases}\text{of interactions} \\ \text{c.f. Newtonian gravity}\end{cases}$
2. Successfully combines two pillars of modern physics:
   a) quantum mechanics
   b) special relativity
   $\Big\}$ together imply particle nonconservation
3. Explains why all fundamental particles of a given type are really, truly identical → they are all excitations of the same underlying field

These are the basic degrees of freedom of our Universe: "operator-valued functions of spacetime"

Units and scales

Three relevant fundamental units

- length, time, and mass
- better are $\hbar$, c, G – linked to fundamental theories [QM, SR, and GR]

  unit of action / unit of velocity

- use natural units $\hbar = c = 1$

  usually eV

  => everything expressed in mass (energy) units

  $[\text{length}] = [\text{time}] = [\text{mass}]^{-1}$

Note "Planck units" are $\hbar = c = G = k_e = k_B = 1$

Useful particle physics conversion $1 \simeq 197$ MeV fm

$\Delta E \gtrsim \frac{\hbar c}{L}$ and $\Delta E \gtrsim 2mc^2$

Two important scales

1. Compton wavelength $\lambda_c = \frac{\hbar}{mc}$

   Distance at which particle-antiparticle pairs become important, for a particle of mass m.

   concept of a pointlike particle breaks down

2. de Broglie wavelength $\lambda_B = \frac{h}{|\vec{p}|}$

   Distance at which particles behave like waves.

Note $\lambda_c < \lambda_B$.

Planck scale $= \sqrt{\frac{\hbar c}{G}} \sim 10^{19}$ GeV

$n, p \sim 1$ GeV

$\Lambda_{\text{cosmo}} \sim 10^{-3}$ eV

Observable Universe $\sim 10^{-33}$ eV

Classical mechanics and quantisation

Let's start by considering a single nonrelativistic particle, and denote its position as $q(t)$

A classical state is specified by coordinates and momenta

We introduce

- Lagrangian $L = \frac{1}{2} m \dot{q}^2 - V(q)$
- Canonical ("conjugate") momentum $p \equiv \frac{\partial L}{\partial \dot{q}} = m\dot{q}$
- Hamiltonian (defined via Legendre transformation)

$$H \equiv p\dot{q} - L = \frac{1}{2} m \dot{q}^2 + V(q)$$

H conserved iff $\frac{\partial L}{\partial t} = 0$

Dynamics are then described by

- Principle of least action (Hamilton's principle) $\delta S = 0$

$\Rightarrow$ Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \Rightarrow \underbrace{m\ddot{q} = -\frac{\partial V}{\partial q}}_{\text{Newton II}}$

- Hamilton's equations

$$\left.\begin{aligned} \dot{q} &= \{q, H\} \\ \dot{p} &= \{p, H\} \end{aligned}\right\} \quad \{q_i, p_j\} = \delta_{ij}$$

N.B. $\{A, B\}_{q,p} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$

Quantisation is now "straightforward" in the Hamiltonian picture

- we treat position and momentum as operators $q, p \to \hat{q}, \hat{p}$
- and impose (equal-time) commutation relations $[\hat{q}, \hat{p}] = i$

N.B. $[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q}$

A quantum state $|\psi\rangle$ specifies a single particle. Observables are operators in a Hilbert space.

Unfortunately the Hamiltonian picture is manifestly non-covariant. This makes the Lagrangian picture more useful for QFT (see 722).

Heisenberg picture

- operators evolve in time according to

$$\frac{d\hat{q}}{dt} = i\,[\hat{H},\hat{q}] = \hat{p}$$

$$\frac{d\hat{p}}{dt} = i\,[\hat{H},\hat{p}] = -\frac{\partial V}{\partial \hat{q}}$$

$$\frac{d\hat{O}}{dt} = i[\hat{H},\hat{O}]$$

- if $\frac{\partial \hat{H}}{\partial t} = 0$ then $\hat{O}(\hat{p},\hat{q};t) = e^{i\hat{H}t}\,O(\hat{p},\hat{q};t=0)\,e^{-i\hat{H}t}$

Now consider the generalisation of a single nonrelativistic particle to a <u>classical field</u>

- imagine a particle at each point on a grid $r_i(t)$
  labels grid points
  $\Rightarrow L = \sum_i \frac{1}{2} m \dot{r}_i^2 + V(r_i)$ etc.

- obtain a field in the <u>continuum limit</u>

$(i, t) \rightarrow x^\mu$ continuous spacetime
$r_i(t) \rightarrow \phi(x^\mu)$ continuous scalar field

- introduce the Lagrange density

$$L \rightarrow \int d^3x\; \mathcal{L}(\phi, \partial_\mu \phi)$$

defined via $(\dot{q}_i(t),\, q_{i+1}(t) - q_i(t)) \rightarrow \frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi$

- equations of motion given by the Euler-Lagrange equations

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

We can also construct the Hamiltonian

- define the "canonical momentum" $\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot\phi}$
- and the Hamiltonian density $\mathcal{H} \equiv \pi(x)\dot\phi(x) - \mathcal{L}$

__Example__

Scalar field theory: a single, real-valued scalar defined at every point in spacetime

Defined by the Lagrange density $\mathcal{L} = \frac{1}{2}\underbrace{\partial_\mu\phi\cdot\partial^\mu\phi}_{\dot\phi^2 - \bar\nabla\phi\cdot\bar\nabla\phi} - \frac{1}{2}m^2\phi^2$

Equations of motion are $(\partial^2 + m^2)\phi(x) = 0$

Klein-Gordon equation

Note: $\partial^2 \equiv \partial_\mu\partial^\mu = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \frac{\partial^2}{\partial t^2} - \bar\nabla^2$

$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \bar\nabla\right) \qquad \partial^\mu = \left(\frac{\partial}{\partial x^0}, -\bar\nabla\right)$

Also: $\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} = \frac{\partial}{\partial(\partial^\mu\phi)}\left(\frac{1}{2}g_{\sigma\nu}\partial^\sigma\phi\,\partial^\nu\phi\right)$
$= \frac{1}{2}g_{\sigma\nu}\delta^{\mu\sigma}\partial^\nu\phi + \frac{1}{2}g_{\sigma\nu}\partial^\sigma\phi\,\delta^{\mu\nu} = \partial^\mu\phi$

Plane wave solutions to the equations of motion are

$$\phi(x) = Ae^{-ip_\mu x^\mu} = Ae^{-i(p^0x^0 - \bar p\cdot\bar x)} = Ae^{-i(Et - \bar p\cdot\bar x)}$$

$\Rightarrow E^2 - |\bar p|^2 = m^2$ or $p^2 = m^2$ ← relativistic dispersion relation

N.B. $(\partial_t^2 + |\bar p|^2 + m^2)\phi(\bar p, t) = 0$ is the E.of M. for an SHO with frequency $\omega_p = \sqrt{|\bar p|^2 + m^2}$

Canonical momentum in this case is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi$$

So the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\bar\nabla\phi\cdot\bar\nabla\phi + \frac{1}{2}m^2\phi$$

Note Hamiltonian formulation non-covariant as $t, \bar x$ not treated equally.

How do we quantise this classical field?

Employ a similar strategy as before

- consider a particle at every point on a spatial grid
- quantise each particle $[\hat{q}_i(t), \hat{p}_j(t)] = i\delta_{ij}$

  particles at different points do not know about quantisation of other particles
- take the continuum limit

$$\boxed{\begin{aligned}&[\hat{\phi}(\bar{x},t), \hat{\pi}(\bar{y},t)] = i\delta^{(3)}(\bar{x}-\bar{y}) \\ &[\hat{\phi}(\bar{x},t), \hat{\phi}(\bar{y},t)] = [\hat{\pi}(\bar{x},t), \hat{\pi}(\bar{y},t)] = 0\end{aligned}}$$

Equal-time commutation relations

Time evolution of operator functionals given by

$$\frac{\partial}{\partial t}\hat{O}(\hat{\phi},\hat{\pi}) = i[\hat{H},\hat{O}]$$

Note: we are in the Heisenberg picture. P+S sec. 2.3 deals with the Schrödinger picture and P+S sec 2.4 treats the Heisenberg picture

Example

Recall our scalar field Lagrange density $\mathcal{L} = \frac{1}{2}\partial_\mu\hat{\phi}\cdot\partial^\mu\hat{\phi} - \frac{1}{2}m^2\hat{\phi}^2$

Then the Hamiltonian density is $\hat{\mathcal{H}} = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\bar{\nabla}\hat{\phi}\cdot\bar{\nabla}\hat{\phi} + \frac{1}{2}m^2\hat{\phi}^2$

$\Rightarrow$ Hamiltonian is $\hat{H} = \int d^3x \left(\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\hat{\phi}(-\bar{\nabla}^2 + m^2)\hat{\phi}\right)\right)$ ← integrated by parts

Equation of motion is now $\left(\frac{\partial^2}{\partial t^2} - \bar{\nabla}^2 + m^2\right)\hat{\phi}(x) = 0$

HOMEWORK

We can solve the operator Klein-Gordon equation in a similar spirit as before, but now we must introduce operator coefficients

$$\hat{\phi}(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left[e^{-i(E_k t - \bar{k}\cdot\bar{x})} \hat{a}(\bar{k}) + e^{+i(E_k t - \bar{k}\cdot\bar{x})} \hat{a}^+(\bar{k}) \right]$$

$E_k = \sqrt{|\bar{k}|^2 + m^2}$ is reminiscent of relativity whereas $\omega_k = \sqrt{|\bar{k}|^2+m^2}$ reminds us this is a SHO

here a, a^+ required to make $\hat{\phi}(x,t)$ real.

Similarly,

$$\hat{\pi}(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} (-iE_k) \left[e^{-i(E_k t - \bar{k}\cdot\bar{x})} \hat{a}(\bar{k}) - e^{+i(E_k t - \bar{k}\cdot\bar{x})} \hat{a}^+(\bar{k}) \right]$$

The commutation relations for $\hat{\phi}, \hat{\pi}$ then imply

HOMEWORK

$$\boxed{\left[\hat{a}(\bar{k}), \hat{a}^+(\bar{p}) \right] = (2\pi)^3 \delta^{(3)}(\bar{k}-\bar{p})}$$

← evaluated at equal-time t, which I drop

Before we explore this solution and its consequences in more detail, let's recall one further aspect of classical mechanics.

SUMMARY

Noether's theorem

my favorite theorem in physics!

Recall Noether's theorem

> Every continuous symmetry of the Lagrangian generates a conserved current.

A continuous symmetry here means any continuous transformation

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \alpha\, \delta\phi(x)$$

← infinitesimal $\alpha \in \mathbb{R}$

that leaves the equations of motion invariant, which, in turn is ensured by invariance of the Lagrangian up to a four-divergence

$$\mathcal{L}(\phi(x)) \longrightarrow \mathcal{L}(\phi(x)) + \alpha\, \partial_\mu J^\mu(\phi(x))$$

Then the conserved current is

$$j^\mu(\phi(x)) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - J^\mu$$

Proof: $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \alpha \delta \mathcal{L}$

$$\alpha \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}(\alpha \delta \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\alpha \delta \phi)$$

$$= \alpha \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi\right) + \alpha \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right]}_{=0 \text{ by E-L eqns.}} \delta\phi$$

where conservation is expressed as

$$\partial_\mu j^\mu(\phi(x)) = 0.$$

$$\frac{\partial j^0}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

Note:

- a conserved current $\Rightarrow$ a conserved charge

$$Q = \int_{R^3} d^3x \, j^0$$

$$\frac{dQ}{dt} = \int_{R^3} d^3\bar{x} \frac{dj^0}{dt} = -\int_{R^3} d^3\bar{x} \, \vec{\nabla} \cdot \vec{j} = 0$$

for $j \to 0$ as $|\bar{x}| \to \infty$

- but a conserved current is a stronger constraint, because it implies charge is conserved locally.

consider charge in a finite volume and apply Stokes' law

$$\frac{dQ_V}{dt} = \int_V d^3\bar{x} \frac{dj^0}{dt} = -\int_V d^3\bar{x} \, \vec{\nabla} \cdot \vec{j} = -\int_A \vec{j} \cdot d\vec{S}$$

Example

Consider massless free scalar field theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2$$

strictly speaking, this would be a global, not local, symmetry

Invariant under an analogue of chiral symmetry $\quad \phi(x) \to \phi'(x) = \phi(x) + \alpha$

relevant to QCD

α a constant

This symmetry is broken by the presence of a mass term

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2$$

$$m^2\phi^2 \to m^2\phi^2 + 2\alpha m^2 \phi + m^2 \alpha^2$$

Example

Consider the infinitesimal spacetime translation $x^\mu \to x^\mu - a^\mu$

We can cast this as a field transformation

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

This transformation behavior is true for any scalar

$$\Rightarrow \mathcal{L}(\phi(x)) \to \mathcal{L}(\phi(x+a)) = \mathcal{L}(\phi(x)) + a^\mu \partial_\mu \mathcal{L}(\phi(x))$$
$$= \mathcal{L}(\phi(x)) + a^\nu \partial_\mu \underbrace{(\delta^\mu{}_\nu \mathcal{L}(\phi(x)))}_{\text{"}J^\mu\text{"}}$$

ie. our 4-divergence from before

In fact four conserved currents

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L}$$

This is the <u>energy-momentum</u> (or stress-energy) <u>tensor</u>

- $H = \int T^{00} d^3\bar{x} = \int \mathcal{H} d^3\bar{x}$ is conserved charge

  invariance under time translation generates energy conservation

- $P^i = \int T^{0i} d^3\bar{x} = - \int \hat{\pi} \partial_i \phi \, d^3\bar{x}$

  invariance under spatial translations generates momentum conservation

Do not confuse physical momentum carried by the field (P^i) with the canonical momentum (π)

strictly this is the momentum density

Scalar fields as quantum harmonic oscillators

We now return to quantum field theory.

Recall we decomposed $\hat{\phi}$ and $\hat{\pi}$ in terms of $\hat{a}^{+}, \hat{a}$

– let's explore that further

$[\hat{a}(k), \hat{a}^{+}(p)] = (2\pi)^3 \delta^{(3)}(\bar{k}-\bar{p})$

c.f. QM $\underbrace{[\hat{a}, \hat{a}^{+}] = 1}$

"ladder" or "raising/lowering" operators

Recall $\hat{H}^{(QM)}_{SHO} = \omega(\hat{a}^{+}\hat{a} + \frac{1}{2})$
has eigenstates $|n\rangle \equiv (a^{+})^{n}|0\rangle$
with eigenvalues $(n+\frac{1}{2})\omega$ and
$|0\rangle$ defined by $\hat{a}|0\rangle \equiv 0$.

In this basis the Hamiltonian is particularly simple

$$
\begin{aligned}
\hat{H} &= \int d^3\bar{x}\ \hat{\mathcal{H}} \\
&= \int d^3\bar{x}\ \frac{1}{2}\left(\hat{\pi}^2(\bar{x}) + \bar{\nabla}\hat{\phi}\cdot\bar{\nabla}\hat{\phi}(\bar{x}) + m^2\hat{\phi}^2(\bar{x})\right) \\
&= \int \frac{d^3\bar{k}}{(2\pi)^3}\,\omega_k\left(\hat{a}^{+}(\bar{k})\hat{a}(\bar{k}) + \frac{1}{2}[\hat{a}(\bar{k}), \hat{a}^{+}(\bar{k})]\right)
\end{aligned}
$$

P.T.O. for proof

Some comments:

- different momenta are not coupled ⇒ we're "diagonalised"
- $[\hat{a}(\bar{k}), \hat{a}^{+}(\bar{k})] = (2\pi)^3\delta^{(3)}(\bar{k}-\bar{k}) = (2\pi)^3\delta^{(3)}(\bar{0}) \sim \infty$ (!!)
  ⇒ this appears to be an infinite contribution to the energy
- physicists generally deal with this problem by ignoring it
  – this "zero point energy" or "vacuum energy" cannot be detected experimentally; only energy differences can be measured

the formal approach to this is "<u>normal ordering</u>"

Or can it ?!
see P+S 22.2 or
Tong's notes 2.3.10
hep-th/0503158

Proof of form of $\hat{H}$

$$\hat{H} = \frac{1}{2}\int d^3\bar{x}\left(\hat{\pi}^2 + (\bar{\nabla}\hat{\phi})^2 + m^2\hat{\phi}^2\right)$$

$$= \frac{1}{2}\int d^3\bar{x}\int\frac{d^3\bar{p}\,d^3\bar{q}}{(2\pi)^6}\Big\{-\frac{\sqrt{E_p E_q}}{2}\left(\hat{a}(\bar{p})e^{i\bar{p}\cdot\bar{x}} - \hat{a}^+(\bar{p})e^{-i\bar{p}\cdot\bar{x}}\right)$$
$$\times\left(\hat{a}(\bar{q})e^{i\bar{q}\cdot\bar{x}} - \hat{a}^+(\bar{q})e^{-i\bar{q}\cdot\bar{x}}\right)$$
$$+\frac{1}{2\sqrt{E_p E_q}}\left(i\bar{p}\hat{a}(\bar{p})e^{i\bar{p}\cdot\bar{x}} - i\bar{p}\hat{a}^+(\bar{p})e^{-i\bar{p}\cdot\bar{x}}\right)$$
$$\times\left(i\bar{q}\hat{a}(\bar{q})e^{i\bar{q}\cdot\bar{x}} - i\bar{q}\hat{a}^+(\bar{q})e^{-i\bar{q}\cdot\bar{x}}\right)$$
$$+\frac{m^2}{2\sqrt{E_p E_q}}\left(\hat{a}(\bar{p})e^{i\bar{p}\cdot\bar{x}} + \hat{a}^+(\bar{p})e^{-i\bar{p}\cdot\bar{x}}\right)\left(\hat{a}(\bar{q})e^{i\bar{q}\cdot\bar{x}} + \hat{a}^+(\bar{q})e^{-i\bar{q}\cdot\bar{x}}\right)\Big\}$$

Let's consider one such term

$$\frac{1}{2}\int d^3\bar{x}\int\frac{d^3\bar{p}\,d^3\bar{q}}{(2\pi)^6}\left(-\frac{1}{2}\right)\sqrt{E_p E_q}\,\hat{a}(\bar{p})\hat{a}(\bar{q})e^{i(\bar{p}+\bar{q})\cdot\bar{x}}$$

$$= \frac{1}{2}\int\frac{d^3\bar{p}\,d^3\bar{q}}{(2\pi)^6}\left(-\frac{1}{2}\right)\sqrt{E_p E_q}\,\hat{a}(\bar{p})\hat{a}(\bar{q})\,\delta^{(3)}(\bar{p}+\bar{q})(2\pi)^3 \qquad \text{N.B. } E_{(-p)} = E_p$$

$$= \frac{1}{4}\int\frac{d^3\bar{p}}{(2\pi)^3}(-E_p)\,\hat{a}(\bar{p})\hat{a}(-\bar{p})$$

Repeating for all the terms we get

$$= \frac{1}{4}\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{E_p}\Big\{\overbrace{(-E_p^2 + \bar{p}^2 + m^2)}^{=0 \text{ since } E_p = \sqrt{|\bar{p}|^2 + m^2}}\left(\hat{a}(\bar{p})\hat{a}(-\bar{p}) + \hat{a}^+(\bar{p})\hat{a}^+(-\bar{p})\right)$$
$$+\underbrace{(E_p^2 + \bar{p}^2 + m^2)}_{2E_p^2}\left(\hat{a}(\bar{p})\hat{a}^+(\bar{p}) + \hat{a}^+(\bar{p})\hat{a}(\bar{p})\right)\Big\}$$

$$= \frac{1}{4}\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{2E_p^2}{E_p}\left(\hat{a}(\bar{p})\hat{a}^+(\bar{p}) + \hat{a}^+(\bar{p})\hat{a}(\bar{p})\right)$$

$$= \frac{1}{2}\int\frac{d^3\bar{p}}{(2\pi)^3}E_p\left(\hat{a}^+(\bar{p})\hat{a}(\bar{p}) + [\hat{a}(\bar{p}),\hat{a}^+(\bar{p})] + \hat{a}^+(\bar{p})\hat{a}(\bar{p})\right)$$

$$= \int\frac{d^3\bar{p}}{(2\pi)^3}E_p\left(\hat{a}^+(\bar{p})\hat{a}(\bar{p}) + \frac{1}{2}[\hat{a}(\bar{p}),\hat{a}^+(\bar{p})]\right)$$

#

Vacuum energies and normal ordering

Let's explore this infinity, its consequences, and how to deal with it, in more detail.

First we define the vacuum state as $|0\rangle$

$$\hat{a}(\bar{k})|0\rangle = 0 \quad \text{for all } \bar{k}$$

Note: if $\hat{H} = \int \frac{d^3\bar{k}}{(2\pi)^3} \omega_k \hat{a}^\dagger(\bar{k})\hat{a}(\bar{k})$ then $[\hat{H}, \hat{a}^\dagger(\bar{k})] = \omega_k \hat{a}^\dagger(\bar{k})$, $[\hat{H}, \hat{a}(\bar{k})] = -\omega_k \hat{a}(\bar{k})$ which is the algebra of an SHO.

Then, writing the Hamiltonian as

$$\hat{H} = \int \frac{d^3\bar{k}}{(2\pi)^3} \omega_k \left(\hat{a}^\dagger(\bar{k})\hat{a}(\bar{k}) + \frac{1}{2}(2\pi)^3 \delta^{(3)}(0) \right) |0\rangle$$

we see the ground state energy comes from the second term

$$\langle 0|\hat{H}|0\rangle = \langle 0| \int \frac{d^3\bar{k}}{(2\pi)^3} \omega_k \frac{1}{2}(2\pi)^3 \delta^{(3)}(0) |0\rangle$$

$$= \langle 0|0\rangle \frac{1}{2}(2\pi)^3 \delta^{(3)}(0) \int \frac{d^3\bar{k}}{(2\pi)^3} \omega_k \equiv E_0$$

In fact there are two infinities in E_0

- $\delta^{(3)}(0)$ arises because we are integrating over all space ← an infrared divergence

  $\Rightarrow$ consider a finite spacetime and take $L \to \infty$

  $$(2\pi)^3 \delta^{(3)}(0) = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3\bar{x}\, e^{i\bar{x}\cdot\bar{k}} \Big|_{\bar{k}=0} = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3\bar{x} = V$$

  We can remove this by considering the ground state energy density

  $$\mathcal{E}_0 = \frac{E_0}{V} = \frac{1}{2} \int \frac{d^3\bar{k}}{(2\pi)^3} \omega_k$$

- But this is still infinite! It has an ultraviolet divergence that arises because we integrate over arbitrarily high momenta. In other words, we assume our theory is valid to arbitrarily small scales, which is clearly false. We can remove this divergence by introducing a UV cutoff.

Rather than "dropping" the irrelevant infinity, we can achieve the same result by the formal procedure of "normal ordering".

Normal ordering : place all annihilation operators to the right of all creation operators

Example

$$: \hat{a}(\bar{k})\hat{a}^+(\bar{k}) + \hat{a}^+(\bar{k})\hat{a}(\bar{k}): = \hat{a}^+(\bar{k})\hat{a}(\bar{k}) + \hat{a}^+(\bar{k})\hat{a}(\bar{k}) = 2\hat{a}^+(\bar{k})\hat{a}(\bar{k})$$

$$: [\hat{a}(\bar{k}), \hat{a}^+(\bar{k})]: = : \hat{a}(\bar{k})\hat{a}^+(\bar{k}) - \hat{a}^+(\bar{k})\hat{a}(\bar{k}): = 0$$

Thus we normal order our Hamiltonian

$$:\hat{H}: = \int \frac{d^3\bar{k}}{(2\pi)^3} \omega_k \hat{a}^+(\bar{k})\hat{a}(\bar{k})$$

and this is equivalent to "dropping" the vacuum energy.

"Particles"

Having dealt with the troublesome infinity via normal ordering, we have a system described by

$$\hat{H} = \int \frac{d^3\bar{k}}{(2\pi)^3} \omega_k \hat{a}^+(\bar{k})\hat{a}(\bar{k}), \quad [\hat{H}, \hat{a}^+(\bar{k})] = \omega_k \hat{a}^+(\bar{k})$$
$$[\hat{H}, \hat{a}(\bar{k})] = \omega_k \hat{a}(\bar{k})$$

I will generally drop the ": :" and assume normal ordering.

This is an SHO for each momentum $\bar{p}$.

Therefore the energy spectrum is given by

$\hat{a}(\bar{k})|0\rangle = 0$ ← vacuum

$\hat{a}^+(\bar{k})|0\rangle = |\bar{k}\rangle$ ← one particle with energy $\hat{H}|\bar{k}\rangle = \omega_k |0\rangle$, $\omega_k = \sqrt{\bar{k}^2 + m^2}$

$\vdots$

$\Rightarrow$ interpret state $|\bar{k}\rangle$ as a "particle" with momentum $\bar{k}$.

Is this reasonable?