Quantum Field Theory I: PHYS 721
In-person session 4: Solutions

Question 1

Show that momentum operator for a free real scalar field

$$\widehat{P}^i = -\int \mathrm{d}^3\vec{x}\ :\widehat{\pi}(\vec{x},t)\nabla^i\widehat{\phi}(\vec{x},t):,$$

can be written in the form

$$\widehat{P}^i = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} k^i \widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}).$$

Solution 1

We write the fields in terms of the ladder operators

$$
\begin{aligned}
\widehat{P}^i &= -\int \mathrm{d}^3\vec{x}\ :\widehat{\pi}(\vec{x},t)\nabla^i\widehat{\phi}(\vec{x},t): \\
&= -\int \mathrm{d}^3\vec{x}\ :\left[i\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\sqrt{\frac{E_k}{2}}\left(e^{-ik_\mu x^\mu}\widehat{a}(\vec{k}) - e^{ik_\mu x^\mu}\widehat{a}^\dagger(\vec{k})\right)\right] \\
&\qquad \times \left[\nabla^i \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(e^{-ip_\mu x^\mu}\widehat{a}(\vec{p}) + e^{ip_\mu x^\mu}\widehat{a}^\dagger(\vec{p})\right)\right]: \\
&= -\frac{i}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\sqrt{\frac{E_k}{E_p}}\int \mathrm{d}^3\vec{x}\ :\left(e^{-ik_\mu x^\mu}\widehat{a}(\vec{k}) - e^{ik_\mu x^\mu}\widehat{a}^\dagger(\vec{k})\right) \\
&\qquad \times \left((-ip^i)e^{-ip^\mu x_\mu}\widehat{a}(\vec{p}) + (ip^i)e^{ip^\mu x_\mu}\widehat{a}^\dagger(\vec{p})\right): \\
&= -\frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\sqrt{\frac{E_k}{E_p}}\,p^i :\Big(e^{-i(E_k+E_p)t}(2\pi)^3\delta^{(3)}(\vec{k}+\vec{p})\widehat{a}(\vec{k})\widehat{a}(\vec{p}) \\
&\qquad - e^{-i(E_k-E_p)t}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{p})\widehat{a}(\vec{k})\widehat{a}^\dagger(\vec{p}) \\
&\qquad - e^{i(E_k-E_p)t}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{p})\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{p}) \\
&\qquad + e^{i(E_k+E_p)t}(2\pi)^3\delta^{(3)}(\vec{k}+\vec{p})\widehat{a}^\dagger(\vec{k})\widehat{a}^\dagger(\vec{p})\Big): \\
&= -\frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\,k^i\left(e^{-2iE_kt}\widehat{a}(\vec{k})\widehat{a}(-\vec{k}) - 2\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}) + e^{2iE_kt}\widehat{a}^\dagger(\vec{k})\widehat{a}^\dagger(-\vec{k})\right) \\
&= -\frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\,k^i\left(2\,\mathrm{Re}\left\{e^{-2iE_kt}\widehat{a}(\vec{k})\widehat{a}(-\vec{k})\right\} - 2\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k})\right).
\end{aligned}
$$

Now

$$
\mathrm{Re}\left\{k^i e^{-2iE_kt}\widehat{a}(\vec{k})\widehat{a}(-\vec{k})\right\} = k^i\cos(2E_kt)\widehat{a}(\vec{k})\widehat{a}(-\vec{k}) \tag{1}
$$

so the first term is odd under $\vec{k} \to -\vec{k}$ and the integral of this term vanishes. Thus we have

$$
\widehat{P}^i = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\,k^i\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}).
$$