Quantum Field Theory I: PHYS 721
In-person session 3: Solutions

Question 1 [Peskin & Schroeder 2.3]

Evaluate the function

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}$$

for $(x-y)$ spacelike so that $(x-y)^2 = -r^2$, explicity in terms of Bessel functions.

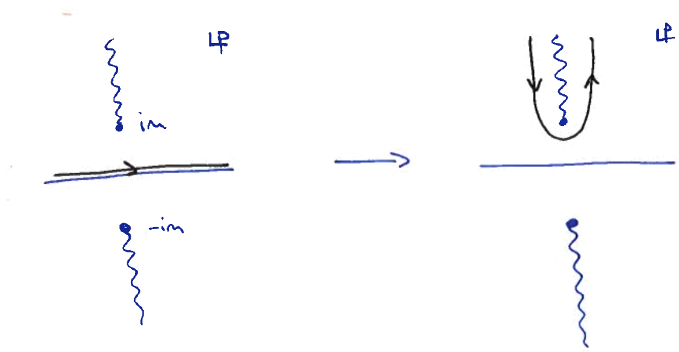
Solution 2

We write the function as

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\sqrt{p^2+m^2}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\
&= \frac{1}{2(2\pi)^3}\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}\mathrm{d}(\cos\theta)\int_0^\infty \mathrm{d}p\frac{p^2}{\sqrt{p^2+m^2}}e^{ipr\cos\theta} \\
&= \frac{2\pi}{2(2\pi)^3}\int_0^\infty \mathrm{d}p\frac{p^2}{\sqrt{p^2+m^2}}\left[\frac{e^{ipr}-e^{-ipr}}{ipr}\right] \\
&= \frac{-i}{2(2\pi)^2\,r}\int_{-\infty}^\infty \mathrm{d}p\frac{p}{\sqrt{p^2+m^2}}e^{ipr},
\end{aligned}$$

where in the last line we've used the variable transform $p \to -p$. Now we need to deform the contour as shown in the figure, which leads us to

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \frac{-i}{2(2\pi)^2\,r}\int_{-\infty}^\infty \mathrm{d}p\frac{p}{\sqrt{p^2+m^2}}e^{ipr} \\
&\to \frac{-i}{2(2\pi)^2\,r}\int_{i\infty}^{-i\infty}\mathrm{d}(i\rho)\frac{(i\rho)}{\sqrt{(i\rho)^2+m^2}}e^{i(i\rho)r} \\
&\to \frac{1}{(2\pi)^2 r}\int_m^\infty \mathrm{d}\rho\frac{\rho}{\sqrt{\rho^2-m^2}}e^{-\rho r} \\
&= \frac{m}{(2\pi)^2 r}K_1(mr).
\end{aligned}$$



Here we've used $\rho = -ip$ and $K_1(z)$ is the modified Bessel function of the second kind, which is the solution to

$$z^2 \frac{\mathrm{d}^2 y(z)}{\mathrm{d}z^2} + z \frac{\mathrm{d}y(z)}{\mathrm{d}z} - (z^2 + 1) y(z) = 0.$$

For large separations, we have

$$K_1(mr) \sim e^{-mr},$$

so that spacelike correlations between massive scalar fields decay exponential with their separation.

Question: What happens in the case of massless scalar fields?