## Quantum Field Theory I: PHYS 721
## Thursday week 1: Solutions

Thursday came and went without a reason
But that's the kind of thing that Thursdays will often do

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### Question 1 [Peskin & Schroeder 2.1]

Classical electromagnetism (with no sources) follows from the action

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F^{\mu\nu}F_{\mu\nu}$$

where $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$.

(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A^\mu(x)$ as the dynamical variables. Write the equations in the standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

(b) Find the equation of motion, which goes by the name of the "Proca equation", for the massive vector field

$$S_m = \int \mathrm{d}^4x \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m_0^2}{2}A_\mu A^\mu\right],$$

and show that

$$\partial_\mu A^\mu(x) = 0$$

if $m_0 > 0$. This is actually a model for the heavy gauge bosons of the weak nuclear force, the $W^\pm$ and $Z^0$ bosons. Why are there only three such bosons and not four (after all, there are four components of $A^\mu$)?

### Solution 1

(a) We derive the equations of motion from

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\right) - \frac{\delta\mathcal{L}}{\delta A_\nu} = 0, \tag{1}$$

where $S = \int \mathrm{d}^4x\mathcal{L}(x)$.

Note that the first term is

$$\begin{aligned}
\partial_\mu \left( \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} \right) &= -\frac{1}{4}\partial_\mu \left( \frac{\delta F^{\rho\sigma} F_{\rho\sigma}}{\delta(\partial_\mu A_\nu)} \right) = -\frac{1}{4}\partial_\mu \left( \frac{\delta F^{\rho\sigma}}{\delta(\partial_\mu A_\nu)} F_{\rho\sigma} + F_{\rho\sigma} \frac{\delta F^{\rho\sigma}}{\delta(\partial_\mu A_\nu)} \right) \\
&= -\frac{1}{4}\partial_\mu \left( \left[\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\nu\rho}\delta^{\mu\sigma}\right] F_{\rho\sigma} + F^{\rho\sigma} \left[\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma\right] \right) \\
&= -\frac{1}{4}\partial_\mu \left( F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} \right) = \partial_\mu F^{\mu\nu}
\end{aligned}$$

and the second is

$$\frac{\delta \mathcal{L}}{\delta A_\nu} = 0.$$

So we obtain

$$\partial_\mu F^{\mu\nu} = 0.$$

Plugging in $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$, we have

$$\partial_i F^{i0} = \partial_i E^i = 0, \tag{2}$$

$$\partial_0 F^{0i} + \partial_j F^{ji} = -\partial_0 E^i + \partial_j \left( -\epsilon^{jik} B^k \right) = \epsilon^{ijk} \partial_j B^k - \partial_0 E^i = 0. \tag{3}$$

This actually only gives two of Maxwell's equations and the other two follow from the Bianchi identity

$$\partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\mu\rho} = 0.$$

These are obtained by noticing that the left hand side of this equation is totally antisymmetric in the indices, so either one is temporal and the other two are spatial, or all three are spatial. Let's choose $\mu = 0$, $\nu = i$, and $\rho = j$, so that we have

$$\partial^j F^{0i} + \partial^0 F^{ij} + \partial^i F^{0j} = \partial^j(-E^i) + \partial^0 \left( -\epsilon^{jik} B^k \right) + \partial^i E^j = 0.$$

We now contract with $\epsilon_{ijl}$ to give

$$\epsilon^{ijl} \partial^i E^j - \epsilon^{ijl} \partial^j E^i + \epsilon^{ijl} \epsilon^{ijk} \partial^0 B^k = 2\epsilon^{ijl} \partial^i E^j + 2\delta^{kl} \partial^0 B^k = \epsilon^{ijl} \partial^i E^j + \partial^0 B^l = 0,$$

where we've used $\epsilon_{ijm}\epsilon_{klm} = \delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}$, so that $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$.

If we take $\mu = i$, $\nu = j$, and $\rho = k$, then we have

$$\partial^k F^{ij} + \partial^i F^{jk} + \partial^j F^{ik} = 0.$$

This we contract with $\epsilon^{ijk}$ to obtain

$$3\epsilon^{ijk} \partial^i F^{jk} = 3\epsilon^{ijk} \left( -\epsilon^{jkl} B^l \right) = -6\delta^{il} B^l = 0.$$

(b) For a derivation that requires material covered later in the course, see arXiv:0901.3300.

But for us, we can start with our previous result and add in

$$\frac{m_0^2}{2}\partial_\mu\left(\frac{\delta A^\rho A_\rho}{\delta(\partial_\mu A_\nu)}\right) = -\frac{1}{4}\partial_\mu\left(\frac{\delta F^{\rho\sigma}}{\delta(\partial_\mu A_\nu)}F_{\rho\sigma} + F_{\rho\sigma}\frac{\delta F^{\rho\sigma}}{\delta(\partial_\mu A_\nu)}\right) = 0$$

and

$$\frac{m_0^2}{2}\left(\frac{\delta A^\rho A_\rho}{\delta A_\nu}\right) = \frac{m_0^2}{2}\left[\delta^{\rho\nu}A_\rho + A^\rho\delta^\nu_\rho\right] = m_0^2 A^\nu.$$

This leads us to the equation of motion

$$\partial_\mu F^{\mu\nu} + m_0^2 A^\nu = 0.$$

If we take a partial derivative, we have

$$\partial_\nu\left[\partial_\mu F^{\mu\nu} + m_0^2 A^\nu\right] = \partial_\nu\partial_\mu F^{\mu\nu} + m_0^2\partial_\nu A^\nu = m_0^2\partial_\nu A^\nu = 0,$$

from which we deduce the constraint

$$\partial_\nu A^\nu = 0.$$

Here we used $\partial_\mu\partial_\nu F^{\mu\nu} = 0$, because of the antisymmetry properties of the field strength tensor.

(c) There are three degrees of freedom because we can use our constraint to eliminate one component of the vector field, say $A^0(x)$,

$$\partial_i\partial^i A^0 + m^2 A^0 = \partial_i\partial^0 A^i.$$

Introducing the mass removes one degree of freedom from our field.