Quantum Field Theory I: PHYS 721
Problem Set 9: Solutions

Chris Monahan

Overview

The questions in this problem set will explore some properties of QED.

Question 1 [8]

Gauge invariance is a strong constraint in field theories, and manifests in many different ways. One manifestation of gauge invariance is the existence of Ward identities, which constrain the form of scattering amplitudes.

In general, if a scattering amplitude involves external photons (let's say there are i incoming external photons and j outgoing), so that the invariant matrix element can be written as

$$\mathcal{M}(A+\sum_i \gamma_i \to B+\sum_j \gamma_j) = \epsilon^{r_1}_{\mu_1}(k_1)\cdots\epsilon^{r_i}_{\mu_i}(k_i)\epsilon^{r_{i+1}*}_{\mu_{i+1}}(k_{i+1})\cdots\epsilon^{r_n*}_{\mu_n}(k_n)\mathcal{M}^{\mu_1\cdots\mu_n}(k_1,\cdots,k_n),$$

where $n=i+j$, then the Ward identity implies

$$k^{\mu_a}\mathcal{M}_{\mu_1\cdots\mu_n}(k_1,\cdots,k_n) = 0, \tag{1}$$

for any $a \in \{1,n\}$.

(a) Consider the case of Compton scattering. The leading order contributions are given by

$$i\mathcal{M}^{(1)} = (-ig)^2\epsilon^*_\mu(k_2)\epsilon_\nu(k_1)\overline{u}(p_2)\gamma^\mu\frac{i(\not{p}_1+\not{k}_1+m)}{(p_1+k_1)^2-m^2}\gamma^\nu u(p_1),$$
$$i\mathcal{M}^{(2)} = (-ig)^2\epsilon^*_\mu(k_2)\epsilon_\nu(k_1)\overline{u}(p_2)\gamma^\nu\frac{i(\not{p}_1-\not{k}_2+m)}{(p_1-k_2)^2-m^2}\gamma^\mu u(p_1).$$

Show that individually the terms do not satisfy the Ward identity,

$$k_2^\mu\mathcal{M}^{(1)}_{\mu\nu} \neq 0 \qquad \text{and} \qquad k_2^\mu\mathcal{M}^{(2)}_{\mu\nu} \neq 0,$$

but the sum does:

$$k_2^\mu\left(\mathcal{M}^{(1)}_{\mu\nu}+\mathcal{M}^{(2)}_{\mu\nu}\right) = 0.$$

Note, you should not need more than about a page of algebra! If you find yourself three pages in, you have probably not chosen the most efficient way to do this!

Solution 1

(a) First we note that the kinematics of Compton scattering satisfy

$$\begin{aligned} p_1 \cdot k_1 &= p_2 \cdot k_2, \\ p_1 \cdot k_2 &= p_2 \cdot k_1, \\ p_1 + k_1 &= p_2 + k_2. \end{aligned}$$

Then we note that the denominators in the propagators are

$$\begin{aligned} (p_1 + k_1)^2 - m^2 &= p_1^2 + k_1^2 + 2p_1 \cdot k_1 = 2p_1 \cdot k_1, \\ (p_1 - k_2)^2 - m^2 &= p_1^2 + k_2^2 + 2p_1 \cdot k_2 = -2p_1 \cdot k_2, \end{aligned}$$

since $p_1^2 - m^2 = 0$ and $k_i^2 = 0$.

Now we examine the first numerator, contracted with the external momentum

$$\begin{aligned} (k_2)_\mu \overline{u}(p_2)\gamma^\mu(\not{p}_1 + \not{k}_1 + m)\gamma^\nu u(p_1) &= \overline{u}(p_2)\not{k}_2(\not{p}_2 + \not{k}_2 + m)\gamma^\nu u(p_1) \\ &= \overline{u}(p_2)(-\not{p}_2\not{k}_2 + 2p_2 \cdot k_2 + k_2^2 + m\not{k}_2)\gamma^\nu u(p_1) \\ &= \overline{u}(p_2)(-m\not{k}_2 + 2p_2 \cdot k_2 + m\not{k}_2)\gamma^\nu u(p_1) \\ &= 2p_2 \cdot k_2 \overline{u}(p_2)\gamma^\nu u(p_1) \\ &= 2p_1 \cdot k_1 \overline{u}(p_2)\gamma^\nu u(p_1), \end{aligned}$$

where in the third line I have used

$$\overline{u}(p_2)(\not{p}_2 + m) = 0,$$

and in the last I have used $p_1 \cdot k_1 = p_2 \cdot k_2$. Thus we have

$$k_2^\mu \mathcal{M}_{\mu\nu}^{(1)} = -ig^2 \frac{2p_1 \cdot k_1}{2p_1 \cdot k_1} \overline{u}(p_2)\gamma^\nu u(p_1) = -ig^2 \overline{u}(p_2)\gamma^\nu u(p_1) \neq 0.$$

Applying the same logic to the second contribution, we have

$$\begin{aligned} (k_2)_\mu \overline{u}(p_2)\gamma^\nu(\not{p}_1 - \not{k}_2 + m)\gamma^\mu u(p_1) &= \overline{u}(p_2)\gamma^\nu(\not{p}_1 - \not{k}_2 + m)\not{k}_2 u(p_1) \\ &= \overline{u}(p_2)\gamma^\nu(-2\not{k}_2\not{p}_1 + 2p_1 \cdot k_2 - k_2^2 + m\not{k}_2)u(p_1) \\ &= \overline{u}(p_2)\gamma^\nu(-2\not{k}_2 m + 2p_1 \cdot k_2 + m\not{k}_2)u(p_1) \\ &= 2p_1 \cdot k_2 \overline{u}(p_2)\gamma^\nu u(p_1), \end{aligned}$$

where I've used

$$(\not{p}_1 - m)u(p_1) = 0.$$

So the second term is

$$k_2^\mu \mathcal{M}^{(2)}_{\mu\nu} = -ig^2 \frac{2p_1 \cdot k_2}{(-2p_1 \cdot k_2)} \overline{u}(p_2)\gamma^\nu u(p_1) = ig^2 \overline{u}(p_2)\gamma^\nu u(p_1) \neq 0.$$

The sum of these terms is

$$k_2^\mu \left[\mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(2)}_{\mu\nu}\right] = -ig^2 \overline{u}(p_2)\gamma^\nu u(p_1) + ig^2 \overline{u}(p_2)\gamma^\nu u(p_1) = 0.$$

Question 2 **[12]**

(a) Evaluate the one-fermion-loop contribution to the photon one-point function, $\langle\Omega|A_\mu|\Omega\rangle$, and show that it vanishes. You shouldn't need to explicitly carry out the integral to show that the result is zero.

(b) Now consider the photon three-point function, $\langle\Omega|A_\mu A_\nu A_\rho|\Omega\rangle$, and write down the two diagrams that contribute to this vacuum expectation value. Show that individually these contributions is nonzero, but that their sum is zero.

(c) By considering the properties under charge transformation of photon n-point functions, show that these functions vanish if n is odd. Note that this result is true nonperturbatively and you should not need to assume a perturbative expansion.

Solution 2

I show the relevant diagrams for the one- and three-point functions of the photon in QED in Figure 1.

(a) The one-point function is proportional to

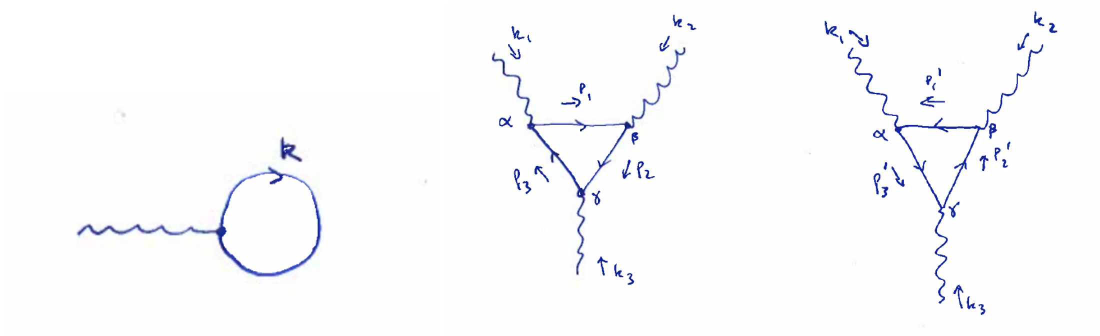
$$\begin{aligned}
\operatorname{Tr} \gamma^\mu \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} &= \operatorname{Tr} \gamma^\mu \gamma^\nu \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\nu}{k^2 - m^2 + i\epsilon} \\
&= 4g^{\mu\nu} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\nu}{k^2 - m^2 + i\epsilon} \\
&= 4 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu}{k^2 - m^2 + i\epsilon}.
\end{aligned}$$

But this integral is odd under $k \to -k$, so it vanishes.

(b) The first contribution to the three-point function is proportional to

$$I_1 = \operatorname{Tr}\left(\gamma^\alpha \frac{\not{p}_1 + m}{p_1^2 - m^2 + i\epsilon} \gamma^\beta \frac{\not{p}_2 + m}{p_2^2 - m^2 + i\epsilon} \gamma^\gamma \frac{\not{p}_3 + m}{p_3^2 - m^2 + i\epsilon}\right),$$

Figure 1: Feynman diagrams that contribute to the photon one- and three-point functions in QED.



where momentum conservation ensures that

$$\begin{aligned} p_1 + k_2 &= p_2, \\ p_2 + k_3 &= p_3, \\ p_3 + k_1 &= p_1. \end{aligned}$$

The nonzero terms include exactly either zero or two factors of the mass, as the other contributions include a trace over an odd number of gamma matrices. These remaining terms are nonzero in general.

The second diagram is given by

$$I_2 = \mathrm{Tr}\left(\gamma^\alpha \frac{\not{q}_3 + m}{q_3^2 - m^2 + i\epsilon} \gamma^\gamma \frac{\not{q}_2 + m}{q_2^2 - m^2 + i\epsilon} \gamma^\beta \frac{\not{q}_1 + m}{q_1^2 - m^2 + i\epsilon}\right),$$

where now we have

$$\begin{aligned} q_2 + k_2 &= q_1, \\ q_3 + k_3 &= q_2, \\ q_1 + k_1 &= q_3. \end{aligned}$$

But clearly $q_i = -p_i$, so we deduce

$$\begin{aligned} p_2 &= p_1 + k_2, \\ p_3 &= p_2 + k_3, \\ p_1 &= p_3 + k_1. \end{aligned}$$

These are exactly the momentum conservation relations we had for the first diagram. Thus we can write

$$\begin{aligned}
I_3 =& \operatorname{Tr}\left(\gamma^\alpha \frac{-\not{p}_3 + m}{p_3^2 - m^2 + i\epsilon}\gamma^\gamma \frac{-\not{p}_2 + m}{p_2^2 - m^2 + i\epsilon}\gamma^\beta \frac{-\not{p}_1 + m}{p_1^2 - m^2 + i\epsilon}\right) \\
=& -\operatorname{Tr}\left(\frac{\not{p}_3 - m}{p_3^2 - m^2 + i\epsilon}\gamma^\gamma \frac{\not{p}_2 - m}{p_2^2 - m^2 + i\epsilon}\gamma^\beta \frac{\not{p}_1 - m}{p_1^2 - m^2 + i\epsilon}\gamma^\alpha\right) \\
=& -\operatorname{Tr}\left(\gamma^\alpha \frac{\not{p}_1 - m}{p_1^2 - m^2 + i\epsilon}\gamma^\beta \frac{\not{p}_2 - m}{p_2^2 - m^2 + i\epsilon}\gamma^\gamma \frac{\not{p}_3 - m}{p_3^2 - m^2 + i\epsilon}\right)
\end{aligned}$$

Again the only nonvanishing terms include zero or two powers of the mass. These terms have the opposite sign to the corresponding terms in I_1, so the sum vanishes.

(c) The interaction term in QED, which couples photons and fermions, is proportional to $\overline{\psi}\gamma^\mu A_\mu \psi$. We already know that charge conjugation acts on the vector bilinear as

$$\widehat{C}\overline{\psi}\gamma^\mu\psi\widehat{C}^{-1} = -\overline{\psi}\gamma^\mu\psi.$$

For QED to be charge-conjugation invariant, we must also have

$$\widehat{C}A_\mu\widehat{C}^{-1} = -A_\mu.$$

Applying this logic to the photon n-point function, Γ_n, we have

$$\begin{aligned}
\Gamma_n &= \langle\Omega|A_{\mu_1}\cdots A_{\mu_n}|\Omega\rangle \\
&= \langle\Omega|A_{\mu_1}\widehat{C}^{-1}\widehat{C}\cdots\widehat{C}^{-1}\widehat{C}A_{\mu_n}\widehat{C}^{-1}\widehat{C}|\Omega\rangle \\
&= \langle\Omega|\widehat{C}A_{\mu_1}\widehat{C}^{-1}\widehat{C}\cdots\widehat{C}^{-1}\widehat{C}A_{\mu_n}\widehat{C}^{-1}|\Omega\rangle \\
&= (-1)^n\langle\Omega|A_{\mu_1}\cdots A_{\mu_n}|\Omega\rangle,
\end{aligned}$$

where I've used

$$\widehat{C}|\Omega\rangle = |\Omega\rangle.$$

Thus we get

$$\Gamma_n = (-1)^n\Gamma_n$$

under charge conjugation. So $\Gamma_n = 0$ for n odd.