Quantum Field Theory I: PHYS 721
Problem Set 8: Solutions

Chris Monahan

Overview

The question in this problem set will explore the effects of an external magnetic field on the relativistic energy levels in hydrogen.

Question [20]

Consider an electron in a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, with $B > 0$ and $\hat{\mathbf{z}}$ a unit vector in the z-direction. The gauge potential is defined through $\mathbf{B} = \nabla \times \mathbf{A}$, so we can take $A^\mu(x^\mu) = (0, 0, Bx, 0)$, where $x^\mu = (0, \mathbf{x}) = (0, x, 0, 0)$.

(a) Write the fermion field in terms of two two-component spinors, $\phi(x^\mu)$ and $\chi(x^\mu)$, as

$$\Psi(x^\mu) = \begin{pmatrix} \phi(x^\mu) \\ \chi(x^\mu) \end{pmatrix}$$

and use the principle of minimal coupling to show that these two-spinors satisfy the Dirac equations,

$$(i\partial_0 - m)\phi(x^\mu) = \sigma \cdot (\mathbf{p} - e\mathbf{A})\chi(x^\mu), \qquad \text{and} \qquad (i\partial_0 + m)\chi(x^\mu) = \sigma \cdot (\mathbf{p} - e\mathbf{A})\phi(x^\mu),$$

where $\mathbf{p} = -i\nabla$ and we assume the Dirac basis for the gamma matrices.

(b) Assume plane-wave solutions of the form

$$\phi(x) = \phi(\mathbf{x})e^{-iEt} \qquad \text{and} \qquad \chi(x) = \chi(\mathbf{x})e^{-iEt}$$

to derive coupled equations for the spatial dependence of both two-spinors. Eliminate the dependence on $\chi(\mathbf{x})$ to obtain a single differential equation for $\phi(\mathbf{x})$.

(c) The components of the momentum p^y and p^z commute with x, so we can search for a solution of the form

$$\phi(\mathbf{x}) = e^{i(p^y y + p^z z)}\nu(x),$$

where $\nu(x)$ is a two-spinor. Furthermore, we can assume that the electron has spin in the z-direction, so that $\nu(x)$ is an eigenfunction of σ_z with eigenvalue $\lambda = \pm 1$. Show that $\nu(x)$

satisfies

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{2}\cdot 2e^2B^2\left(x-\frac{p^y}{eB}\right)^2\right]\nu(x)=\left(E^2-m^2-(p^z)^2+\lambda eB\right)\nu(x)$$

(d) This equation is formally identical to the Schrödinger equation for a well-studied quantum mechanical system. Identify the system and the relevant physical parameter that characterises this system. Use this to identify the energy levels of the system and show that they can be written as

$$E(n,p^z,\lambda)=\sqrt{m^2+(p^z)^2+(2n+1+\lambda)|e|B},$$

where the charge of the electron is $e=-|e|$. Identify the degeneracies in this system.

(e) Show that this result for the energy levels reduces to the usual nonrelativistic (Landau) levels for a spin-1/2 particle in an external magnetic field, in the nonrelativistic limit $(p^z)^2 \ll m^2$ and $(2n+1)|e|B \ll m^2$.

Solution

(a) We start from the Dirac equation for the four-spinor

$$(i\not{D}-m)\Psi(x^\mu)=0,$$

and substitute in the Dirac representation for the gamma matrices, and the spinor representation required for the question

$$\begin{aligned}
[i\partial_\mu\gamma^\mu-e\gamma^\mu A_\mu-m]\begin{pmatrix}\phi(x^\mu)\\ \chi(x^\mu)\end{pmatrix}&=\\
\left[i(\partial_0\gamma^0+\partial_i\gamma^i)-e\gamma^i A_i-m\right]\begin{pmatrix}\phi(x^\mu)\\ \chi(x^\mu)\end{pmatrix}&=\\
\left[i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\partial_0+i\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\partial_i+\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}eA_i-m\right]\begin{pmatrix}\phi(x^\mu)\\ \chi(x^\mu)\end{pmatrix}&=0.
\end{aligned}$$

Now we split this into two two-by-two equations

$$\begin{aligned}
(i\partial_0-m)\phi(x^\mu)&=-\sigma\cdot(i\nabla+e\mathbf{A})\chi(x^\mu)\\
(-i\partial_0-m)\chi(x^\mu)&=\sigma\cdot(i\nabla+e\mathbf{A})\phi(x^\mu),
\end{aligned}$$

and substitute $\mathbf{p} = -i\nabla$ to obtain

$$(i\partial_0 - m)\phi(x^\mu) = \sigma \cdot (\mathbf{p} - e\mathbf{A})\chi(x^\mu)$$
$$(i\partial_0 + m)\chi(x^\mu) = \sigma \cdot (-\mathbf{p} - e\mathbf{A})\phi(x^\mu),$$

as required.

(b) Plugging in

$$\phi(x) = \phi(\mathbf{x})e^{-iEt}, \qquad \text{and} \qquad \chi(x) = \chi(\mathbf{x})e^{-iEt},$$

we find

$$(E - m)\phi(\mathbf{x}) = \sigma \cdot (\mathbf{p} - e\mathbf{A})\chi(\mathbf{x})$$
$$(E + m)\chi(\mathbf{x}) = \sigma \cdot (\mathbf{p} - e\mathbf{A})\phi(\mathbf{x}).$$

Rearranging the second of these for $\chi(\mathbf{x})$, we have

$$\chi(\mathbf{x}) = \frac{1}{E+m}\sigma \cdot (\mathbf{p} - e\mathbf{A})\phi(\mathbf{x}),$$

which we substitute into the first equation to give

$$\begin{aligned}(E - m)\phi(\mathbf{x}) &= \frac{1}{E+m}\left[\sigma \cdot (\mathbf{p} - e\mathbf{A})\right]^2 \phi(\mathbf{x}) \\ &= \frac{1}{E+m}\sigma^i\sigma^j(p^i - eA^i)(p^j - eA^j)\phi(\mathbf{x}) \\ &= \frac{1}{E+m}\left[\delta^{ij} + i\epsilon^{ijk}\sigma^k\right](p^i - eA^i)(p^j - eA^j)\phi(\mathbf{x}) \\ &= \frac{1}{E+m}\left[(\mathbf{p} - e\mathbf{A})^2 + i\epsilon^{ijk}\sigma^k(-ep^i)A^j\right]\phi(\mathbf{x}).\end{aligned}$$

Now we replace $p^i \to -i\nabla^i$ to write

$$i\epsilon^{ijk}\sigma^k(-ep^i)A^j = i\epsilon^{ijk}(ei\nabla^i)A^j\sigma^k = -eB^k\sigma^k = -e\mathbf{B}\cdot\boldsymbol{\sigma}.$$

Putting this all together, we find

$$(E^2 - m^2)\phi(\mathbf{x}) = \left[(\mathbf{p} - e\mathbf{A})^2 - e\mathbf{B}\cdot\boldsymbol{\sigma}\right]\phi(\mathbf{x}) = \left[\mathbf{p}^2 + e^2B^2x^2 - 2ep^yBx - e\sigma^zB\right]\phi(\mathbf{x}).$$

Here we have used $[p^i, A^j] = 0$ for our choice of gauge field, $A^y = Bx$, and $B^i = (0, 0, B)$.

(c) At this stage, we can apply the Ansatz in the question

$$\phi(\mathbf{x}) = e^{i(p^y y + p^z z)}\nu(x),$$

which leads to, with the substitution $p^i = (-i\partial_x, p^y, p^z)$,

$$(E^2 - m^2)e^{i(p^y y + p^z z)}\nu(x) = \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (p^y)^2 + (p^z)^2 + e^2B^2x^2 - 2ep^yBx - e\sigma^z B\right] e^{i(p^y y + p^z z)}\nu(x).$$

This holds for arbitrary y and z, so we deduce

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (p^y - eBx)^2 - e\sigma^z B\right]\nu(x) = (E^2 - m^2 - (p^z)^2)\nu(x).$$

Now we assume that $\nu(x)$ is an eigenfunction of σ^z, so we have

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (p^y - eBx)^2 - e\lambda B\right]\nu(x) = (E^2 - m^2 - (p^z)^2)\nu(x).$$

Rearranging this, we obtain the requred result

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\tilde{x}^2} + \frac{1}{2}\cdot 2e^2B^2\left(x - \frac{p^y}{eB}\right)^2\right]\nu(x) = \left(E^2 - m^2 - (p^z)^2 + \lambda eB\right)\nu(x).$$

(d) This equation correspond to the Schrödinger equation for a one-dimensional simple harmonic oscillator

$$\left[-\frac{1}{2\tilde{m}}\frac{\mathrm{d}^2}{\mathrm{d}\tilde{x}^2} + \frac{1}{2}\tilde{m}\omega^2\tilde{x}^2\right]\nu(x) = E_n\nu(x),$$

where $\tilde{x} = x - p^y/(eB)$, with characteristic frequency $\omega = 2|e|B$ and effective mass $\tilde{m} = 1/2$. The energy levels are given by

$$E_n = \left(n + \frac{1}{2}\right)\omega,$$

or

$$E^2 - m^2 - (p^z)^2 + \lambda eB = \left(n + \frac{1}{2}\right)2|e|B.$$

This is a quadratic equation for the energy levels, which we can solve as

$$E = E(n, p^z, \lambda) = \sqrt{m^2 + (p^z)^2 + (2n + 1 + \lambda)|e|B},$$

(e) There is a continuous degeneracy in p^x and p^y, and a discrete degeneracy between

$$E(n, p^z, \lambda = +1) = E(n + 1, p^z, \lambda = -1). \tag{1}$$

There is also a two-fold degeneracy in $p^z \to -p^z$.

The expansion of the square root is

$$\sqrt{1+x} \simeq 1 + \frac{x}{2} + \dots,$$

so we can expand the energies as

$$E = E(n, p^z, \lambda) = m\sqrt{1 + \frac{(p^z)^2}{m^2} + (2n+1+\lambda)\frac{|e|B}{m}} \simeq m + \frac{(p^z)^2}{2m} + \left(n + \frac{1+\lambda}{2}\right)\frac{|e|B}{2m}.$$

Note that $|e|/(2m)$ is the Bohr magneton, μ_B, so that this reads,

$$E \simeq m + \frac{(p^z)^2}{2m} + \left(n + \frac{1+\lambda}{2}\right)\mu_B B,$$

which is exactly the usual (classical) nonrelativistic expression for the energy of an electron in a magnetic field.