Quantum Field Theory I: PHYS 721
Problem Set 7: Solutions

Chris Monahan

Overview

The questions in this problem set introduce you to the Lorentz-invariant "Mandelstam variables" for two-to-two scattering.

Question 1 [12]

Consider a scattering process $A(p_1)A(p_2) \to A(q_1)A(q_2)$, involving only a single species of particle, A, with mass, m.

The entire process can be considered to occur in a plane, so we only need one angle and one energy variable to describe the process. It is simplest to use the centre-of-momentum frame, in which $\vec{p}_1 + \vec{p}_2 = \vec{q}_1 + \vec{q}_2 = 0$. Define the direction of $\vec{p}_1$ to be the z-axis and the direction of $\vec{q}_1$ to be at an angle $\theta_{\rm CM}$ with respect to the z-axis.

We can define three Lorentz-invariant kinematic variables, which are given by

$$s \equiv (p_1 + p_2)^2, \qquad t \equiv (q_1 - p_1)^2, \qquad \text{and} \qquad u \equiv (q_2 - p_1)^2.$$

These are the Lorentz invariant Mandelstam variables.

(a) Show that these variables are related by the condition

$$s + t + u = 4m^2,$$

so that there are only two independent variables.

(b) Show that these Mandelstam variables can be related to the centre-of-momentum frame variables by

$$\begin{aligned} s &= E_{\rm CM}^2, \\ t &= 2|\vec{p}_{\rm CM}|^2(\cos\theta_{\rm CM} - 1), \\ u &= -2|\vec{p}_{\rm CM}|^2(1 + \cos\theta_{\rm CM}), \end{aligned}$$

where

$$|\vec{p}_{\rm CM}| = \sqrt{\frac{E_{\rm CM}^2}{4} - m^2}$$

is the momentum of any of the particles in the centre-of-momentum frame.

(c) What are the ranges of the Mandelstam variables for physical scattering?

(d) How do you think the relation between the Mandelstam variables changes if the particles have different masses?

Solution 1

(a) In the center-of-momentum frame we have

$$\vec{p}_1 + \vec{p}_2 = \vec{q}_1 + \vec{q}_2 = 0,$$

and we can write the momenta as

$$\begin{aligned}
p_1^\mu &= (E_{\text{CM}}/2, 0, 0, |\vec{p}_{\text{CM}}|) \\
p_2^\mu &= (E_{\text{CM}}/2, 0, 0, -|\vec{p}_{\text{CM}}|) \\
q_1^\mu &= (E_{\text{CM}}/2, |\vec{p}_{\text{CM}}| \sin\theta_{\text{CM}}, 0, |\vec{p}_{\text{CM}}| \cos\theta_{\text{CM}}) \\
q_2^\mu &= (E_{\text{CM}}/2, -|\vec{p}_{\text{CM}}| \sin\theta_{\text{CM}}, 0, -|\vec{p}_{\text{CM}}| \cos\theta_{\text{CM}})
\end{aligned}$$

Then the Mandelstam variables are

$$\begin{aligned}
s &\equiv (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = 2m^2 + 2p_1 \cdot p_2, \\
t &\equiv (q_1 - p_1)^2 = q_1^2 + p_1^2 - 2p_1 \cdot q_1 = 2m^2 - 2p_1 \cdot q_1, \\
u &\equiv (q_2 - p_1)^2 = q_2^2 + p_1^2 - 2p_1 \cdot q_2 = 2m^2 - 2p_1 \cdot q_2.
\end{aligned}$$

The sum of these is then

$$\begin{aligned}
s + t + u &= 2m^2 + 2p_1 \cdot p_2 + 2m^2 - 2p_1 \cdot q_1 + 2m^2 - 2p_1 \cdot q_2 \\
&= 6m^2 + 2p_1 \cdot p_2 - 2p_1 \cdot (q_1 + q_2) \\
&= 6m^2 + 2p_1 \cdot p_2 - 2p_1 \cdot (p_1 + p_2) \\
&= 6m^2 - 2p_1^2 \\
&= 4m^2.
\end{aligned}$$

(b) In the centre-of-momentum frame we have

$$(p_1 + p_2)^\mu = (E_{\text{CM}}, 0, 0, 0), \tag{1}$$
$$(q_1 - p_1)^\mu = (0, |\vec{p}_{\text{CM}}| \sin\theta_{\text{CM}}, 0, \vec{p}_{\text{CM}}(\cos\theta_{\text{CM}} - 1)), \tag{2}$$
$$\tag{3}$$

so that

$$
\begin{aligned}
&s = (p_1 + p_2)^2 = E_{\mathrm{CM}}^2, \\
&t = (q_1 - p_1)^2 = -(|\vec{p}_{\mathrm{CM}}|^2 \sin^2\theta_{\mathrm{CM}} + |\vec{p}_{\mathrm{CM}}|^2(\cos\theta_{\mathrm{CM}} - 1)^2) = 2|\vec{p}_{\mathrm{CM}}|^2(\cos\theta_{\mathrm{CM}} - 1).
\end{aligned}
$$

Then we can most easily calculate u via

$$
\begin{aligned}
u &= 4m^2 - s - t \\
&= 4m^2 - 4(|\vec{p}_{\mathrm{CM}}|^2 + m^2) + 2|\vec{p}_{\mathrm{CM}}|^2(1 - \cos\theta_{\mathrm{CM}}) \\
&= -2|\vec{p}_{\mathrm{CM}}|^2(1 + \cos\theta_{\mathrm{CM}}).
\end{aligned}
$$

(c) The minimum energy in the centre-of-momentum frame is $E_{\mathrm{CM}} = 2m$, so for physical scattering we must have $s \geq 4m^2$. The cosine function is restricted to the range $[-1, 1]$, so $-4|\vec{p}_{\mathrm{CM}}|^2 \leq t, u \leq 0$, although u and t do not have the same endpoints as a function of θ_{CM} (for example $\theta_{\mathrm{CM}} = 0$ means $t = 0$, which is forward scattering, but $u = -4|\vec{p}_{\mathrm{CM}}|^2$).

(d) If the particles have different masses, m_i, then the relationship between the Mandelstam variables becomes

$$
s + t + u = \sum_{i=1}^{4} m_i^2.
$$

Question 2 **[8]**

Recall the scalar Yukawa theory of Homework 5:

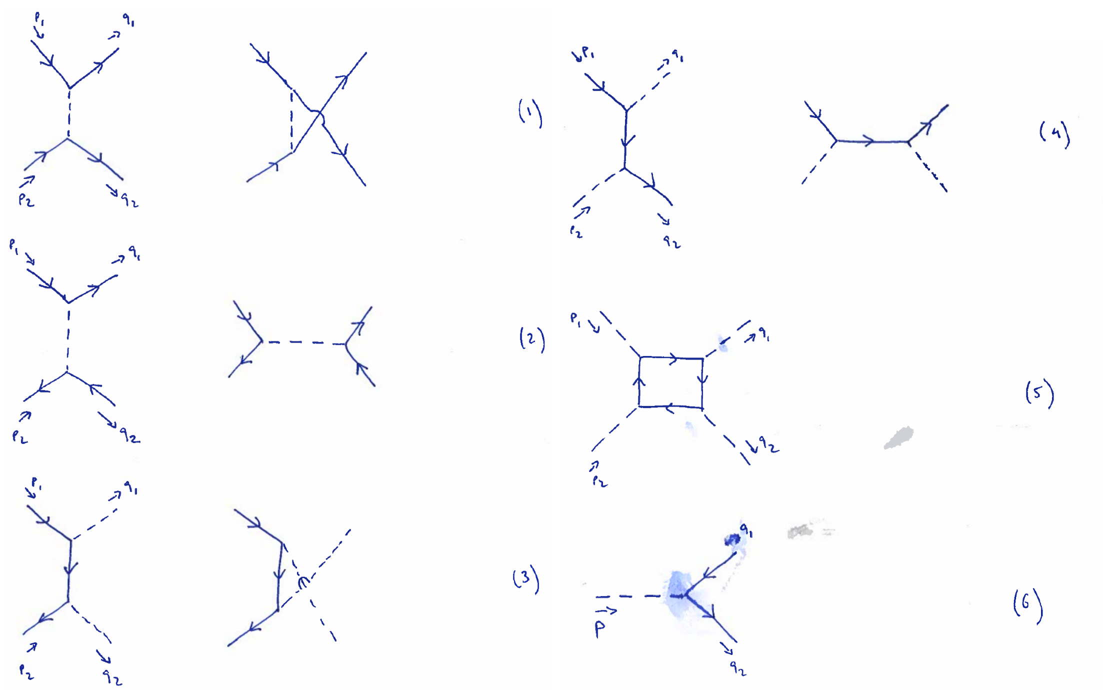
$$
\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - M^2 \psi^* \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - g \psi^* \psi \phi.
$$

Here ψ is a complex scalar field and ϕ is a real scalar field.

Draw the leading nontrivial (i.e. do not count the case of no scattering) Feynman diagrams for the following processes, and express the invariant matrix element for each process in terms of Mandelstam variables:

1. Nucleon-nucleon scattering, $\psi\psi \to \psi\psi$;
2. Nucleon-antinucleon scattering, $\psi\psi^* \to \psi\psi^*$;
3. Meson pair production, $\psi\psi^* \to \phi\phi$;
4. Nucleon-meson scattering, $\psi\phi \to \psi\phi$.

Figure 1: Feynman diagrams that contribute to each of the processes listed in the bullet points one to five, labelled by the appropriate number. Solid lines are nucleons and antinucleons and dashed lines are mesons. The momenta in the right hand diagrams are understood to be the same as in the left hand diagrams. [Ignore diagrams (5) and (6).]



Solution 2

(a) The Feynman diagrams for these processes appear in Fig. 1. The invariant matrix elements for these processes are

1. Nucleon-nucleon scattering,

$$i\mathcal{M} = -ig^2 \left[\frac{1}{t - m^2} + \frac{1}{u - m^2} \right];$$

2. Nucleon-antinucleon scattering,

$$i\mathcal{M} = -ig^2 \left[\frac{1}{t - m^2} + \frac{1}{s - m^2 + i\epsilon} \right];$$

3. Meson pair production,

$$i\mathcal{M} = -ig^2 \left[\frac{1}{t - M^2} + \frac{1}{u - M^2} \right];$$

4. Nucleon-meson scattering,

$$i\mathcal{M} = -ig^2 \left[\frac{1}{s - M^2 + i\epsilon} + \frac{1}{u - M^2} \right].$$