Quantum Field Theory I: PHYS 721
Problem Set 6: Solutions

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Overview

The questions in this problem give you some practice at manipulating spinors and gamma matrices.

Question 1 [10]

(a) Prove the spinor relation

$$\overline{u}^r(\vec{p})\gamma^\mu u^s(\vec{q}) = \frac{1}{2m}\overline{u}^r(\vec{p})\left[p^\mu + q^\mu + i\sigma^{\mu\nu}(p_\nu - q_\nu)\right]u^s(\vec{q}).$$

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. This relation is known as the "Gordon identity". Note that this is named after Walter Gordon of Klein-Gordon equation fame, and not Paul Gordan of Clebsch-Gordan coefficient fame.

(b) Using the identity

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta},$$

where $\epsilon_{12} = +1$, show that

$$\left[\overline{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_2\right]\left[\overline{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_4\right] = -\left[\overline{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_4\right]\left[\overline{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_2\right].$$

Here $\overline{u}_{1,3}$ and $u_{2,4}$ are four different spinors. This is an example of a "Fierz identity". These identities relate products of spinor bilinears to sums of products of more useful spinor bilinears.

Both the Gordon and various Fierz identities are often used in calculations of scattering amplitudes involving fermions.

Solution 1

(a) To prove the Gordon identity, we first note that

$$\overline{u}^r(\vec{p})\gamma^\mu(\gamma^\nu q_\nu - m)u^s(\vec{q}) = 0,$$

and that

$$\overline{u}^r(\vec{p})(\gamma^\nu p_\nu - m)\gamma^\mu u^s(\vec{p}) = 0.$$

Adding these together, we obtain

$$2m\overline{u}^r(\vec{p})\gamma^\mu u^s(\vec{q}) = \overline{u}^r(\gamma^\mu\gamma^\nu q_\nu + \gamma^\nu\gamma^\mu p_\nu)u^s(\vec{q}).$$

But we can write (recall lecture 10)

$$\begin{aligned}\gamma^\mu\gamma^\nu &= g^{\mu\nu} - i\sigma^{\mu\nu},\\ \gamma^\nu\gamma^\mu &= g^{\mu\nu} + i\sigma^{\nu\mu},\end{aligned}$$

so we have

$$\begin{aligned}\overline{u}^r(\vec{p})\gamma^\mu u^s(\vec{q}) &= \frac{1}{2m}\overline{u}^r\left[q_\nu\left(g^{\mu\nu} - i\sigma^{\mu\nu}\right) + p_\nu\left(g^{\mu\nu} + i\sigma^{\nu\mu}\right)\right]u^s(\vec{q})\\ &= \frac{1}{2m}\overline{u}^r\left[q^\mu + p^\mu + i\sigma^{\mu\nu}(p_\nu - q_\nu)\right]u^s(\vec{q}).\end{aligned}$$

(b) To prove this particular Fierz identity, we first note that the projection operator serves to isolate the right-handed components of the two spinors u_2 and u_4 (i.e. the corresponding Weyl spinors). Then, in the Weyl basis, the gamma matrix γ^μ are block off-diagonal. Thus the left hand side of this identity reduces to

$$\left[\overline{u}_{1,R}\sigma^\mu u_{2,R}\right]\left[\overline{u}_{3,R}\sigma_\mu u_{4,R}\right].$$

Then, applying the identity given in the question, this becomes

$$\begin{aligned}\left[\overline{u}_{1,R}\sigma^\mu u_{2,R}\right]\left[\overline{u}_{3,R}\sigma_\mu u_{4,R}\right] &= (\overline{u}_{1,R})_\alpha(\sigma^\mu)_{\alpha\beta}(u_{2,R})_\beta(\overline{u}_{3,R})_\gamma(\sigma_\mu)_{\gamma\delta}(u_{4,R})_\delta\\ &= 2\epsilon_{\alpha\gamma}(\overline{u}_{1,R})_\alpha(\overline{u}_{3,R})_\gamma\epsilon_{\beta\delta}(u_{2,R})_\beta(u_{4,R})_\delta\\ &= -2\epsilon_{\alpha\gamma}(\overline{u}_{1,R})_\alpha(\overline{u}_{3,R})_\gamma\epsilon_{\delta\beta}(u_{2,R})_\beta(u_{4,R})_\delta\\ &= -(\overline{u}_{1,R})_\alpha(\sigma^\mu)_{\alpha\delta}(u_{2,R})_\beta(\overline{u}_{3,R})_\gamma(\sigma_\mu)_{\gamma\beta}(u_{4,R})_\delta\\ &= -\left[\overline{u}_{1,R}\sigma^\mu u_{4,R}\right]\left[\overline{u}_{3,R}\sigma_\mu u_{2,R}\right]\\ &= -\left[\overline{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_4\right]\left[\overline{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_2\right]\end{aligned}$$

Question 2 **[10]**

(a) Show that the vector representation of the Lorentz group satisfies the Lorentz Lie algebra, that is, show that

$$[\overset{1}{J}{}^{\mu\nu}, \overset{1}{J}{}^{\rho\sigma}] = i\left(g^{\nu\rho}\,\overset{1}{J}{}^{\mu\sigma} - g^{\mu\rho}\,\overset{1}{J}{}^{\nu\sigma} - g^{\nu\sigma}\,\overset{1}{J}{}^{\mu\rho} + g^{\mu\sigma}\,\overset{1}{J}{}^{\nu\rho}\right).$$

(b) Show the spinor representation of the Lorentz group satisfies the Lorentz Lie algebra, that is, show that

$$[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \overset{\frac{1}{2}}{J}{}^{\rho\sigma}] = i\left(g^{\nu\rho}\ \overset{\frac{1}{2}}{J}{}^{\mu\sigma} - g^{\mu\rho}\ \overset{\frac{1}{2}}{J}{}^{\nu\sigma} - g^{\nu\sigma}\ \overset{\frac{1}{2}}{J}{}^{\mu\rho} + g^{\mu\sigma}\ \overset{\frac{1}{2}}{J}{}^{\nu\rho}\right).$$

Solution 2

(a) For this we need

$$\begin{aligned}(J^{\mu\nu})_{\alpha\gamma}(J^{\rho\sigma})^{\gamma\beta} &= -\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\gamma} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\gamma}\right)\left(\delta^{\rho\gamma}\delta^{\sigma\beta} - \delta^{\rho\beta}\delta^{\sigma\gamma}\right) \\ &= -\delta^{\mu}_{\alpha}\delta^{\rho\nu}\delta^{\sigma\beta} + \delta^{\mu}_{\alpha}\delta^{\rho\beta}\delta^{\sigma\nu} + \delta^{\nu}_{\alpha}\delta^{\rho\mu}\delta^{\sigma\beta} - \delta^{\nu}_{\alpha}\delta^{\rho\beta}\delta^{\sigma\mu}\end{aligned}$$

and

$$\begin{aligned}(J^{\rho\sigma})_{\alpha\gamma}(J^{\mu\nu})^{\gamma\beta} &= -\left(\delta^{\rho}_{\alpha}\delta^{\sigma}_{\gamma} - \delta^{\rho}_{\gamma}\delta^{\sigma}_{\alpha}\right)\left(\delta^{\mu\gamma}\delta^{\nu\beta} - \delta^{\mu\beta}\delta^{\nu\gamma}\right) \\ &= -\delta^{\rho}_{\alpha}\delta^{\sigma\mu}\delta^{\nu\beta} + \delta^{\rho}_{\alpha}\delta^{\mu\beta}\delta^{\sigma\nu} + \delta^{\sigma}_{\alpha}\delta^{\rho\mu}\delta^{\nu\beta} - \delta^{\sigma}_{\alpha}\delta^{\rho\nu}\delta^{\mu\beta}.\end{aligned}$$

Putting these together, we have

$$\begin{aligned}[J^{\mu\nu}, J^{\rho\sigma}]^{\beta}_{\alpha} = \delta^{\nu\rho}&\left(-\delta^{\mu}_{\alpha}\delta^{\sigma\beta} + \delta^{\sigma}_{\alpha}\delta^{\mu\beta}\right) + \delta^{\nu\sigma}\left(\delta^{\mu}_{\alpha}\delta^{\rho\beta} - \delta^{\rho}_{\alpha}\delta^{\mu\beta}\right) \\ &+ \delta^{\mu\rho}\left(\delta^{\nu}_{\alpha}\delta^{\sigma\beta} - \delta^{\sigma}_{\alpha}\delta^{\nu\beta}\right) - \delta^{\mu\sigma}\left(\delta^{\nu}_{\alpha}\delta^{\rho\beta} - \delta^{\rho}_{\alpha}\delta^{\nu\beta}\right) \\ = ig^{\nu\rho}i&\left(\delta^{\mu}_{\alpha}\delta^{\sigma\beta} - \delta^{\sigma}_{\alpha}\delta^{\mu\beta}\right) - ig^{\nu\sigma}i\left(\delta^{\mu}_{\alpha}\delta^{\rho\beta} - \delta^{\rho}_{\alpha}\delta^{\mu\beta}\right) \\ &- ig^{\mu\rho}i\left(\delta^{\nu}_{\alpha}\delta^{\sigma\beta} - \delta^{\sigma}_{\alpha}\delta^{\nu\beta}\right) + ig^{\mu\sigma}i\left(\delta^{\nu}_{\alpha}\delta^{\rho\beta} - \delta^{\rho}_{\alpha}\delta^{\nu\beta}\right) \\ = ig^{\nu\rho}&(J^{\mu\nu})^{\beta}_{\alpha} - ig^{\nu\sigma}(J^{\mu\rho})^{\beta}_{\alpha} - ig^{\mu\rho}(J^{\nu\sigma})^{\beta}_{\alpha} + ig^{\mu\sigma}(J^{\nu\rho})^{\beta}_{\alpha}.\end{aligned}$$

Note that we have some useful properties of the delta function with raised and lowered indices. We start from the definition

$$\delta^{\mu}{}_{\mu} = +1,$$

for all μ (with no summation over μ in this case). Then it follows that

$$\delta^{\mu\nu} = \delta^{\mu}{}_{\rho}g^{\rho\nu} = g^{\mu\nu}$$

In particular we have

$$\begin{aligned}\delta^{00} &= \delta^{0}{}_{0}g^{00} = +1, \\ \delta^{ij} &= \delta^{i}{}_{k}g^{kj} = -1\end{aligned}$$

for $i = j\ (= k)$. All other values are zero.

Similarly we have

$$\delta_{\mu\nu} = \delta^{\rho}{}_{\nu} g_{\rho\mu} = g^{\rho\sigma} g_{\rho\mu} g_{\sigma\nu},$$

where we've used

$$g^{\rho\sigma} g_{\sigma\nu} = \delta^{\rho}{}_{\nu}.$$

Thus

$$\delta_{00} = g^{00} g_{00} g_{00} = +1,$$

and, with no summation over i implied,

$$\delta_{ii} = g^{jk} g_{ij} g_{ik} = (-1)^3 = -1.$$

So we also have

$$\delta_{\mu\nu} = g_{\mu\nu}.$$

(b) To show that spinor representation satisfies the Lie algebra of the Lorentz group, we use the result of (c). Plugging in the definition of the spinor representation, and taking $\rho \neq \sigma$, we have

$$\begin{aligned}
[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \overset{\frac{1}{2}}{J}{}^{\rho\sigma}] &= \frac{i}{2}[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^{\rho}\gamma^{\sigma}] \\
&= \frac{i}{2}\left([\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^{\rho}]\gamma^{\sigma} + \gamma^{\rho}[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^{\sigma}]\right).
\end{aligned}$$

Now we use part (c) to write this as

$$\begin{aligned}
[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \overset{\frac{1}{2}}{J}{}^{\rho\sigma}] &= \frac{i}{2}\left(i\left(\gamma^{\mu} g^{\nu\rho} - \gamma^{\nu} g^{\rho\mu}\right)\gamma^{\sigma} + \gamma^{\rho}\, i\left(\gamma^{\mu} g^{\nu\sigma} - \gamma^{\nu} g^{\sigma\mu}\right)\right) \\
&= \frac{i}{2}\left(i\gamma^{\mu}\gamma^{\sigma} g^{\nu\rho} - i\gamma^{\nu}\gamma^{\sigma} g^{\rho\mu} + i\gamma^{\rho}\gamma^{\mu} g^{\nu\sigma} - \gamma^{\rho}\gamma^{\nu} g^{\sigma\mu}\right).
\end{aligned} \tag{1}$$

Now we recall that

$$\overset{\frac{1}{2}}{J}{}^{\mu\nu} = \frac{i}{2}\gamma^{\mu}\gamma^{\nu} - \frac{i}{2} g^{\mu\nu},$$

or, rearranging this, that

$$i\gamma^{\mu}\gamma^{\nu} = 2\,\overset{\frac{1}{2}}{J}{}^{\mu\nu} + i g^{\mu\nu}.$$

Using this in Equation (1), we have

$$\begin{aligned}
[\overset{\frac{1}{2}}{J}^{\mu\nu}, \overset{\frac{1}{2}}{J}^{\rho\sigma}] &= \frac{i}{2}\Big(\Big(2\overset{\frac{1}{2}}{J}^{\mu\sigma} + ig^{\mu\sigma}\Big)g^{\nu\rho} - i\Big(2\overset{\frac{1}{2}}{J}^{\nu\sigma} + ig^{\nu\sigma}\Big)g^{\rho\mu} \\
&\qquad + i\Big(2\overset{\frac{1}{2}}{J}^{\rho\mu} + ig^{\rho\mu}\Big)g^{\nu\sigma} - \Big(2\overset{\frac{1}{2}}{J}^{\rho\nu} + ig^{\rho\nu}\Big)g^{\sigma\mu}\Big) \\
&= i\Big(\overset{\frac{1}{2}}{J}^{\mu\sigma} g^{\nu\rho} - \overset{\frac{1}{2}}{J}^{\nu\sigma} g^{\rho\mu} - \overset{\frac{1}{2}}{J}^{\rho\nu} g^{\sigma\mu} + \overset{\frac{1}{2}}{J}^{\rho\mu} g^{\nu\sigma} \\
&\qquad + \frac{i}{2}\left(g^{\mu\sigma}g^{\nu\rho} - g^{\nu\sigma}g^{\rho\mu} + g^{\rho\mu}g^{\nu\sigma} - g^{\rho\nu}g^{\sigma\mu}\right) \\
&= i\Big(\overset{\frac{1}{2}}{J}^{\mu\sigma} g^{\nu\rho} - \overset{\frac{1}{2}}{J}^{\nu\sigma} g^{\rho\mu} - \overset{\frac{1}{2}}{J}^{\rho\nu} g^{\sigma\mu} + \overset{\frac{1}{2}}{J}^{\rho\mu} g^{\nu\sigma}\Big).
\end{aligned}$$