Nucleon-nucleon scattering in scalar Yukawa Theory: A Mini-Project

Christopher J. Monahan[1]

[1]*Department of Physics, William & Mary, Williamsburg, Virginia 23187, USA*

Nucleon-nucleon scattering is a central to the experimental understanding of a wide range of physical systems, from nuclei to crystals and biomaterials. I study nucleon-nucleon scattering in a toy model: scalar Yukawa theory at next-to-next-to-leading order in perturbation theory. Working in the interaction picture, I use Wick's theorem to derive an expression for the scattering amplitude up to, and including, terms of $\mathcal{O}(g^4)$. I state the Feynman rules for this theory and discuss the relation of fully-connected Feynman diagrams to expressions derived from Wick's theorem. I conclude by discussing possible extensions of this work, including the more realistic Yukawa model of meson-nucleon interactions, which treats nucleons as fermionic fields.

I. INTRODUCTION

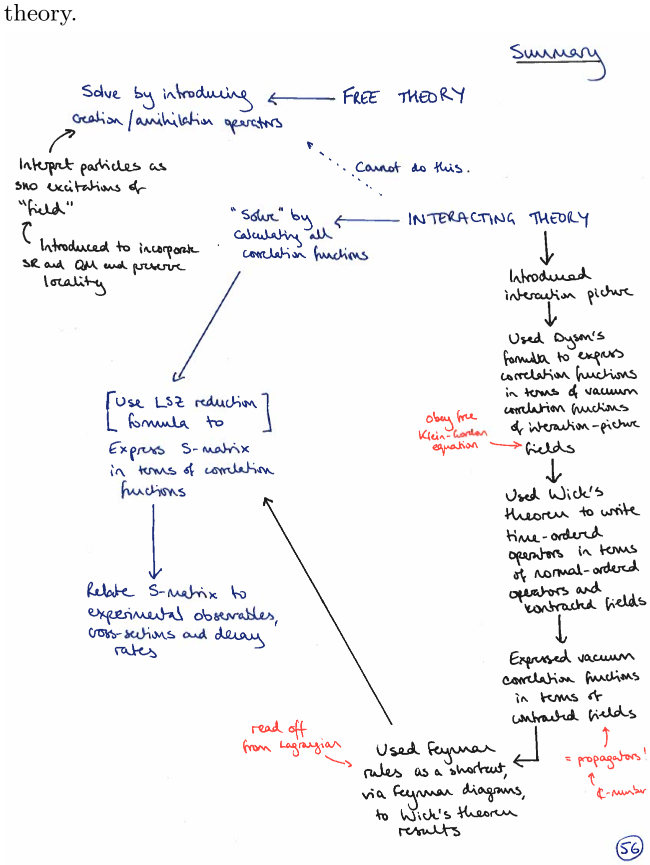
Our Universe is composed of interacting particles, which are described through the mathematical formalism of interacting quantum field theory. In contrast to the theory of free quantum fields, interacting quantum field theory cannot be solved exactly. To circumvent this difficulty, we assume the fully interacting theory is a small perturbation of the corresponding free theory. We introduce the "interaction picture" as a way to treat the fields in the interacting theory as though they were free particles.

We then express all correlation functions of the interacting fields in terms of time-ordered products of the fields in the interaction picture, via Dyson's formula. We use Dyson's series to expand the time evolution operator (the exponential of the interacting piece of the Hamiltonian) in a perturbative expansion. By applying Wick's theorem to the resulting time-ordered products of fields, we can express vacuum matrix elements of these products in terms of vacuum matrix elements of normal-ordered products of fields, which vanish, and all possible Wick-contracted pairs of fields. Each Wick contracted pair of fields corresponds to a propagator. Each non-vanishing diagram corresponds to a term in Wick's theorem that includes only contracted pairs of fields, and no normal-ordered fields. We can construct Feynman diagrams to represent the contributions to the original vacuum matrix elements. These Feynman diagrams are, in turn, expressed in terms of momentum integrals via the Feynman rules, which can be read off from the Lagrangian that defines our theory.

We make contact with experimental observables through the LSZ reduction formula, which relates the Scattering matrix, or "S-matrix", to Feynman diagrams (or, equivalently, correlation functions of our theory). Observables such as scattering cross-sections and decay rates can be expressed in terms of the S-matrix and, through the LSZ reduction formula, to Feynman diagrams. I represent this process schematically in Fig. 1.

As an example of this process, I study the scalar analogue of nucleon-nucleon scattering in the scalar Yukawa model. This toy model exhibits many important features of scattering theory in the context of quantum field theories and provides a natural starting point for understanding nucleon-nucleon scattering in more realistic theories.

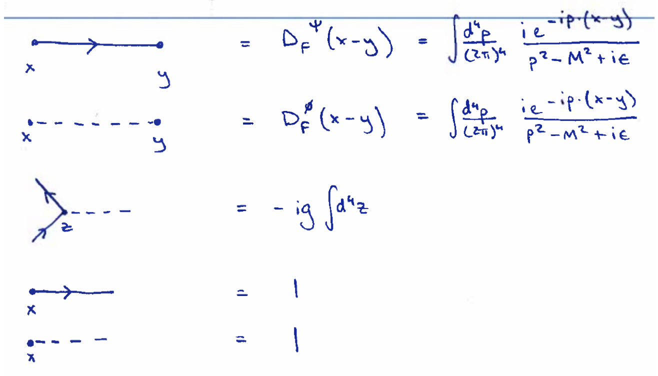
FIG. 1. Schematic representation of scattering in scalar field theory.



II. SCALAR YUKAWA MODEL

In the rest of this mini-project, I study the scalar Yukawa model of nucleon-meson interactions. For brevity, I do not indicate operator-valued field explicitly:

FIG. 2. Feynman rules for the scalar Yukawa theory. The solid lines represent the propagator for the complex field ψ and the dashed line the propagator for the real field ϕ. The vertex represents the interaction between $\psi^\dagger$, ψ, and ϕ. In addition to these graphical elements, each diagram should be divided by the appropriate symmetry factor, and all unconstrained momenta should be integrated.



all fields should be interpreted as operators.

The scalar Yukawa model is defined by the Lagrangian density

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - M^2\psi^*\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi, \tag{1}$$

where ψ is a complex scalar field, corresponding to a scalar nucleon, and ϕ is a real scalar field, corresponding to a meson. I will assume that $g \ll M, m$, so that I can apply perturbative methods to the example of nucleon-nucleon scattering $\psi\psi \to \psi\psi$. The Feynman rules for this theory can be read from this Lagrangian density and are shown in Fig. 2. Throughout this mini-project, I work in the interaction picture and assume all operators are normal ordered.

The Hamiltonian of this theory is defined through

$$H = \int d^3\vec{x}\,\mathcal{H} = \int d^3\vec{x}\left[\sum_{i=1}^{3}\pi^{(i)}\dot{q}^{(i)} - \mathcal{L}\right], \tag{2}$$

where the sum runs over the number of particle species, which in this case is three (ψ, ψ^*, and ϕ). Here

$$\pi^{(i)} = \frac{\partial\mathcal{L}}{\partial\dot{q}^{(i)}} \tag{3}$$

is the conjugate momentum field for $q^{(i)}$. In this model the Hamiltonian is given by

$$H = H_0 + H_I, \tag{4}$$

where H_0 is the free Hamiltonian,

$$H_0 = \int d^3\vec{x}\left(\pi_\psi\pi_\psi^* + \frac{1}{2}\pi_\phi\pi_\phi + \vec{\nabla}\psi^*\cdot\vec{\nabla}\psi + \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + M^2\psi^*\psi + \frac{m^2}{2}\phi^2\right), \tag{5}$$

and H_I is the interaction Hamiltonian,

$$H_I = g\int d^3\vec{x}\,\psi^*\psi\phi. \tag{6}$$

In the interaction picture, in which the fields obey the classical equations of motion for free scalar fields, we can express the fields in terms of creation and annihilation operators

$$\psi(x) = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[b^\dagger(\vec{p})e^{-ip\cdot x} + c(\vec{p})e^{ip\cdot x}\right], \tag{7}$$

$$\psi^\dagger(x) = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[b(\vec{p})e^{ip\cdot x} + c^\dagger(\vec{p})e^{-ip\cdot x}\right], \tag{8}$$

$$\phi(x) = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a^\dagger(\vec{p})e^{-ip\cdot x} + a(\vec{p})e^{ip\cdot x}\right]. \tag{9}$$

The ϕ field creates and annihilates mesons through the $a^\dagger$ and a operators, respectively. Similarly, the ψ field creates scalar nucleons through $b^\dagger$ and annihilates scalar antinucleons through c, while the $\psi^\dagger$ field creates antinucleons via $c^\dagger$ and annihilates nucleons with b.

In terms of these creation and annihilation operators, the normal-ordered Hamiltonian can be decomposed into

$$:H_0: = \frac{1}{2}\int\frac{d\vec{p}}{(2\pi)^3}\Big(a^\dagger(\vec{p})a(\vec{p}) + b^\dagger(\vec{p})b(\vec{p}) + c^\dagger(\vec{p})c(\vec{p})\Big), \tag{10}$$

and

$$\begin{aligned}:H_I: = g\int\frac{d^3\vec{p}\,d^3\vec{q}}{(2\pi)^6}\frac{1}{2\sqrt{2E_pE_q}}\\ \times\Bigg[\frac{1}{\sqrt{E_{p+q}}}\Big(a(-\vec{p}-\vec{q})b(\vec{p})c(\vec{q}) + a^\dagger(-\vec{p}-\vec{q})b^\dagger(\vec{p})c^\dagger(\vec{q})\\ + b^\dagger(\vec{p})c^\dagger(\vec{q})a(\vec{p}+\vec{q}) + a^\dagger(\vec{p}+\vec{q})b(\vec{q})c(\vec{p})\Big)\\ + \frac{1}{\sqrt{E_{p-q}}}\Big(b^\dagger(\vec{p})a(\vec{p}-\vec{q})b(\vec{q}) + c^\dagger(\vec{p})a^\dagger(\vec{p}-\vec{q})c(\vec{q})\Big)\\ + \frac{1}{\sqrt{E_{q-p}}}\Big(b^\dagger(\vec{p})a^\dagger(\vec{q}-\vec{p})b(\vec{q}) + c^\dagger(\vec{q})a(\vec{q}-\vec{p})c(\vec{p})\Big)\Bigg].\end{aligned} \tag{11}$$

III. NUCLEON-NUCLEON SCATTERING

The scattering amplitude for the scalar analogue of nucleon-nucleon scattering is given by

$$\langle f|S|i\rangle = \langle f|1 + iT|i\rangle, \tag{12}$$

where S is the scattering matrix (or "S-matrix") and T, the "T-matrix", represents the nontrivial contribution to this amplitude. The S-matrix is the time-ordered evolution operator of the fields in the interaction picture, in the limit that the initial state is defined in the infinite past and the final state in the infinite future.

In the case of nucleon-nucleon scattering, the initial and final states can be represented as

$$|i\rangle = |\psi\psi\rangle, \tag{13}$$
$$\langle f| = \langle\psi\psi|. \tag{14}$$

These are also known as the "in" and "out" states, respectively. In terms of the creation and annihilation operators of Eqs. (7) to (9), these states are given by

$$|i\rangle \equiv |p_1 p_2\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\, b^\dagger(\vec{p}_1) b^\dagger(\vec{p}_2)|0\rangle, \tag{15}$$
$$\langle f| \equiv \langle q_1 q_2| = \sqrt{2E_{q_1}}\sqrt{2E_{q_2}}\,\langle 0| b(\vec{q}_1) b(\vec{q}_2). \tag{16}$$

Here I have implicitly assumed that the scattering states, the "in" and "out" states, are eigenstates of the free theory. Although this is not strictly true, it provides the foundation for understanding the LSZ formalism, which allows physical states, even in confining theories, to be defined via the poles of external propagators. Implicit in the assumption of free scattering states is the cancellation of disconnected and unamputated diagrams in the scattering amplitude.

A. Dyson's series and Wick's theorem

To calculate the nontrivial part of this scattering amplitude, we need to express the S-matrix in terms of fields in the interaction picture, which we do through the Dyson series

$$U_I(-T,T) = \mathcal{T}\left\{\exp\left[-i\int_{-T}^{T} \mathrm{d}\tau\, H_I(\tau)\right]\right\}, \tag{17}$$

where $\mathcal{T}$ is the time-ordering operator, and we obtain the scattering matrix S by considering the limit $T \to \infty(1 - i\epsilon)$, with $\epsilon \ll 1$. Using our expressions for these fields in terms of creation and annihilation operators, Eqs. (7) to (9), this expression becomes

$$S = \mathcal{T}\left\{\exp\left[-ig\int \mathrm{d}^4 z\, \psi^\dagger\psi\phi\right]\right\}. \tag{18}$$

Therefore, using Eqs. (15) and (16), the time-ordered product of operators that we need is

$$\begin{aligned}
&\mathcal{T}\Big\{b(\vec{q}_1)b(\vec{q}_2)\exp\left[-ig\int \mathrm{d}^4 z\,\psi^\dagger\psi\phi\right] b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\Big\}\\
&= \mathcal{T}\Big\{b(\vec{q}_1)b(\vec{q}_2)\Big[1 - ig\int \mathrm{d}^4 z\,\psi^\dagger\psi\phi\\
&\quad - \frac{g^2}{2}\Big(\int \mathrm{d}^4 z\,\psi^\dagger(z)\psi(z)\phi(z)\Big)^2\\
&\quad + \frac{ig^3}{6}\Big(\int \mathrm{d}^4 z\,\psi^\dagger(z)\psi(z)\phi(z)\Big)^3\\
&\quad + \frac{g^4}{24}\Big(\int \mathrm{d}^4 z\,\psi^\dagger(z)\psi(z)\phi(z)\Big)^4\Big] b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\Big\}.
\end{aligned} \tag{19}$$

We can express this in terms of normal ordered products of fields and contracted pairs of fields, at each order in perturbation theory, using Wick's theorem.

The leading order term, of $\mathcal{O}(g^0)$, gives

$$\begin{aligned}
\mathcal{T}\left\{b(\vec{q}_1)b(\vec{q}_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\right\} &= :b(\vec{q}_1)b(\vec{q}_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\\
&+ b(\vec{q}_1)b(\vec{q}_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2) + b(\vec{q}_1)b(\vec{q}_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\\
&+ b(\vec{q}_1)b(\vec{q}_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2):.
\end{aligned} \tag{20}$$

In this notation, the contraction between operators of different momenta implies a delta function in those momenta, which imposes four momentum conservation. The normal-ordered products of fields will give vanishing contributions to the vacuum matrix element, because each normal-ordered product contains an annihilation operator acting on the free vacuum, which is zero, by definition of the vacuum state.

Disconnected contributions do not contribute to scattering amplitudes, since those cancel when the "in" and "out" states in the interaction picture are connected to the free field vacuum via the interacting vacuum, through Dyson's equation. So in the following, I consider only fully connected contributions.

At higher orders in perturbation theory, there will be similar such expressions, although the next nonvanishing contribution to the vacuum matrix element is at $\mathcal{O}(g^2)$, since that is the first contribution that has two real scalar fields that can be contracted.

Thus, at $\mathcal{O}(g^2)$, the only nonvanishing connected contributions to the scattering amplitude are given by

$$\begin{aligned}
&\mathcal{T}\Big\{b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_1)\psi(z_1)\phi(z_1)\\
&\quad\times \psi^\dagger(z_2)\psi(z_2)\phi(z_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\Big\}\\
&= \Big(b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_1)\psi(z_1)\psi^\dagger(z_2)\psi(z_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\\
&\quad + b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_1)\psi(z_1)\psi^\dagger(z_2)\psi(z_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\\
&\quad + b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_1)\psi(z_1)\psi^\dagger(z_2)\psi(z_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\\
&\quad + b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_1)\psi(z_1)\psi^\dagger(z_2)\psi(z_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\Big)\\
&\quad\times \phi(z_1)\phi(z_2).
\end{aligned} \tag{21}$$

There are no nonvanishing contributions at $\mathcal{O}(g^3)$, because of the presence of an uncontracted real field ϕ. At $\mathcal{O}(g^4)$ there are 144 contractions between the complex scalar fields and three possible contractions between the real scalar fields, although not all of these contractions are fully connected. The connected contributions fall into just four broad classes of diagrams, which I denote $T^{(i)}$, so that we have

$$\begin{aligned}
&\mathcal{T}\Big\{b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_1)\psi(z_1)\phi(z_1)\psi^\dagger(z_2)\psi(z_2)\phi(z_2) \\
&\qquad\times\psi^\dagger(z_3)\psi(z_3)\phi(z_3)\psi^\dagger(z_4)\psi(z_4)\phi(z_4)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\Big\} \\
&\qquad\equiv \sum_{i=1}^{4}\mathcal{O}^{(i)}. \qquad (22)
\end{aligned}$$

Each of these classes, $\mathcal{O}^{(i)}$, appears with its crossed partner (i.e. with $q_1 \leftrightarrow q_2$), as illustrated in Fig. 4.

The first of these classes is

$$\begin{aligned}
\mathcal{O}^{(1)}_{g^4} &= b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_1)\psi^\dagger(z_4)\psi(z_2)\psi^\dagger(z_3)\psi(z_3)\psi^\dagger(z_2) \\
&\quad\times\psi(z_1)\psi(z_4)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4),
\end{aligned} \tag{23}$$

while the second is

$$\begin{aligned}
\mathcal{O}^{(2)}_{g^4} &= b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_1)\psi^\dagger(z_2)\psi(z_1)\psi^\dagger(z_3)\psi(z_2)\psi^\dagger(z_4) \\
&\quad\times\psi(z_3)\psi(z_4)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4).
\end{aligned} \tag{24}$$

These correspond to diagrams (a) and (c) in Fig. 4, respectively.

The third class, represented by diagram (e) of Fig. 4, has the same complex scalar field contractions as the first class, but with different real scalar field contractions:

$$\begin{aligned}
\mathcal{O}^{(3)}_{g^4} &= b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_4)\psi^\dagger(z_4)\psi(z_2)\psi^\dagger(z_3)\psi(z_3)\psi^\dagger(z_2) \\
&\quad\times\psi(z_1)\psi(z_2)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\phi(z_1)\phi(z_4)\phi(z_2)\phi(z_3),
\end{aligned} \tag{25}$$

and the fourth class represents the "vertex corrections" to the meson-nucleon interactions:

$$\begin{aligned}
\mathcal{O}^{(4)}_{g^4} &= b(\vec{q}_1)b(\vec{q}_2)\psi^\dagger(z_3)\psi^\dagger(z_4)\psi(z_1)\psi^\dagger(z_2)\psi(z_2)\psi^\dagger(z_3) \\
&\quad\times\psi(z_1)\psi(z_4)b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\phi(z_1)\phi(z_3)\phi(z_2)\phi(z_4).
\end{aligned} \tag{26}$$

I represent this term by diagram (g) in Fig. 4.

B. Scattering amplitudes

We can now use these results from Wick's theorem to write down expressions for the scattering amplitude for scalar nucleon-nucleon scattering. We first need expressions for the scalar field contractions,

$$\psi(x)\psi^\dagger(y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-M^2+i\epsilon}, \tag{27}$$

$$\phi(x)\phi(y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}. \tag{28}$$

and for the mixed-representation contractions, such as

$$\psi(z_1)b^\dagger(\vec{p}_1) = e^{-ip_1\cdot z_1}, \tag{29}$$

which correspond to contractions of, in this case, the ψ field with an external initial state nucleon. In the notation of Peskin and Schroeder [1], this is represented by

$$\psi(z_1)\psi(z_2)|p_1p_2\rangle = \left(e^{-ip_1\cdot z_1}+e^{-ip_2\cdot z_2}\right)|0\rangle. \tag{30}$$

Let's consider the vacuum matrix element of the first of the terms in Eq. (21), which at $\mathcal{O}(g^2)$ I denote $T^{(1)}_{g^2}$, as an example. This is given by

$$\begin{aligned}
T^{(1)}_{g^2} &= \frac{(ig)^2}{2!}\int \mathrm{d}^4z_1\mathrm{d}^4z_2\, e^{iz_1\cdot q_1}e^{-iz_1\cdot p_1}e^{iz_2\cdot q_2}e^{-iz_2\cdot p_2} \\
&\quad\times\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{ie^{-ik\cdot(z_1-z_2)}}{k^2-m^2+i\epsilon},
\end{aligned} \tag{31}$$

which is the first term in Eq. (3.48) of David Tong's lecture notes [2]. The spatial integrals arise from the time integral over the interaction Hamiltonian, and the prefactor is generated by the expansion of the (time-ordered) exponential of the interaction Hamiltonian.

Carrying out the spatial integrals leads to

$$\begin{aligned}
T^{(1)}_{g^2} &= \frac{(ig)^2}{2!}\int \mathrm{d}^4k\,(2\pi)^4\delta^{(4)}(q_1-p_1-k) \\
&\quad\times\delta^{(4)}(q_2-p_2+k)\frac{i}{k^2-m^2+i\epsilon} \\
&= -\frac{g^2}{2}\int\mathrm{d}^4k\,\frac{i}{(p_2-q_2)^2-m^2+i\epsilon} \\
&\quad\times(2\pi)^4\delta^{(4)}(q_1-p_1+q_2-p_2).
\end{aligned} \tag{32}$$

Applying this procedure to all four terms that appear in Eq. (21) leads to

$$\begin{aligned}
T_{g^2} &= -ig^2\left[\frac{1}{(p_2-q_2)^2-m^2}+\frac{1}{(p_1-q_2)^2-m^2}\right] \\
&\quad\times(2\pi)^4\delta^{(4)}(q_1-p_1+q_2-p_2).
\end{aligned} \tag{33}$$

Here the kinematics of the process dictate that the ϕ meson can never go onshell and we may therefore set $\epsilon = 0$. This is discussed in more detail in [2].

At the next order, we apply the same logic, but the algebra is significantly more involved. I consider just one of the terms in Eq. (22), given by Eq. (23), as an example:

$$
\begin{aligned}
\mathcal{O}^{(1)}_{g^4} &= \frac{(ig)^4}{4!}\int \mathrm{d}^4z_1\cdots \mathrm{d}^4z_4\, e^{iq_1\cdot z_1}e^{iq_2\cdot z_4}e^{-ip_1\cdot z_1}e^{-ip_2\cdot z_4}\\
&\quad\times\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{ie^{-ik_1\cdot(z_2-z_3)}}{k_1^2-M^2+i\epsilon}\frac{ie^{-ik_2\cdot(z_3-z_2)}}{k_4^2-M^2+i\epsilon}\\
&\quad\times\int \frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\frac{ie^{-ik_3\cdot(z_1-z_2)}}{k_3^2-m^2+i\epsilon}\frac{ie^{-ik_4\cdot(z_3-z_4)}}{k_4^2-m^2+i\epsilon}\\
&= \frac{g^4}{4!}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\cdots\frac{\mathrm{d}^4k_4}{(2\pi)^4}\frac{i}{k_1^2-M^2+i\epsilon}\\
&\quad\times\frac{i}{k_4^2-M^2+i\epsilon}\frac{i}{k_3^2-m^2+i\epsilon}\frac{i}{k_4^2-m^2+i\epsilon}\\
&\quad\times(2\pi)^4\delta^{(4)}(q_1-p_1-k_3)(2\pi)^4\delta^{(4)}(k_2-k_1+k_3)\\
&\quad\times(2\pi)^4\delta^{(4)}(k_1-k_2-k_4)(2\pi)^4\delta^{(4)}(q_2-p_2+k_4).
\end{aligned}
\tag{34}
$$

Carrying out these integrals, we have

$$
\begin{aligned}
\mathcal{O}^{(1)}_{g^4} &= \frac{g^4}{4!}\frac{(2\pi)^4}{((p_1-q_1)^2-m^2+i\epsilon)^2}\delta^{(4)}(q_1-p_1+q_2-p_2)\\
&\quad\times\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-M^2+i\epsilon}\frac{1}{(k+p_1-q_1)^2-M^2+i\epsilon}.
\end{aligned}
\tag{35}
$$

The other contributions, $\mathcal{O}^{(2)}_{g^4}$ to $\mathcal{O}^{(4)}_{g^4}$, can be obtained similarly, but it is much simpler to read off the relevant expressions from the Feynman diagrams, as I show below.

C. Feynman perturbation theory

The time ordered expression in Eq. (22), with each class of terms given in Eqs. (23) to (26), is the relevant operator that contributes to the scattering amplitude at $\mathcal{O}(g^4)$, expressed in terms of Wick-contracted fields. Within each class, all connected permutations should be included.

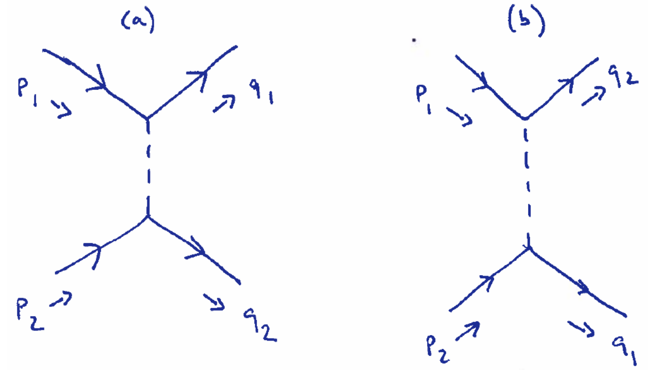
Although Wick's theorem gives the exact result at this order, it is computationally much simpler to represent the scattering amplitude at a given order in perturbation theory through Feynman diagrams. I illustrate the leading contributions to the T-matrix, that is, at $\mathcal{O}(g^2)$, in Fig. 3. Note that there are no $\mathcal{O}(g^0)$ contributions to the T-matrix, as these contributions correspond the case of no scattering.

Using the Feynman rules of Fig. 2, these Feynman diagrams are given in momentum space by

$$
I^{(a)}_{g^2} = -g^2\frac{i}{(q_1-p_1)^2+m^2}(2\pi)^4\delta^{(4)}(q_1-p_1+q_2-p_2),
\tag{36}
$$

$$
I^{(b)}_{g^2} = -g^2\frac{i}{(q_2-p_1)^2+m^2}(2\pi)^4\delta^{(4)}(q_1-p_1+q_2-p_2).
\tag{37}
$$

FIG. 3. Feynman diagrams that contribute to nucleon-nucleon scattering at $\mathcal{O}(g^2)$ in perturbation theory. The solid lines represent the complex scalar field propagator D_F^{ψ} and the dashed lines the real scalar field propagator D_F^{ϕ}. Each interaction vertex is associated with a factor of the coupling constant $-ig$ in momentum space.



The sum of these contributions gives the scattering amplitude, T_{g^2}, in agreement with Eq. (21).

In Fig. 4 I show the Feynman diagrams that contribute to nucleon-nucleon scattering at the next order, $\mathcal{O}(g^4)$.

The diagrams of Fig. 4 represent the following contributions:

$$
\begin{aligned}
I^{(a)}_{g^4} &= g^4\frac{1}{(q_i-p_i)^2+m^2+i\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+M^2+i\epsilon)^2},\\
&\quad\times(2\pi)^4\delta^{(4)}(q_1-p_1+q_2-p_2),
\end{aligned}
\tag{38}
$$

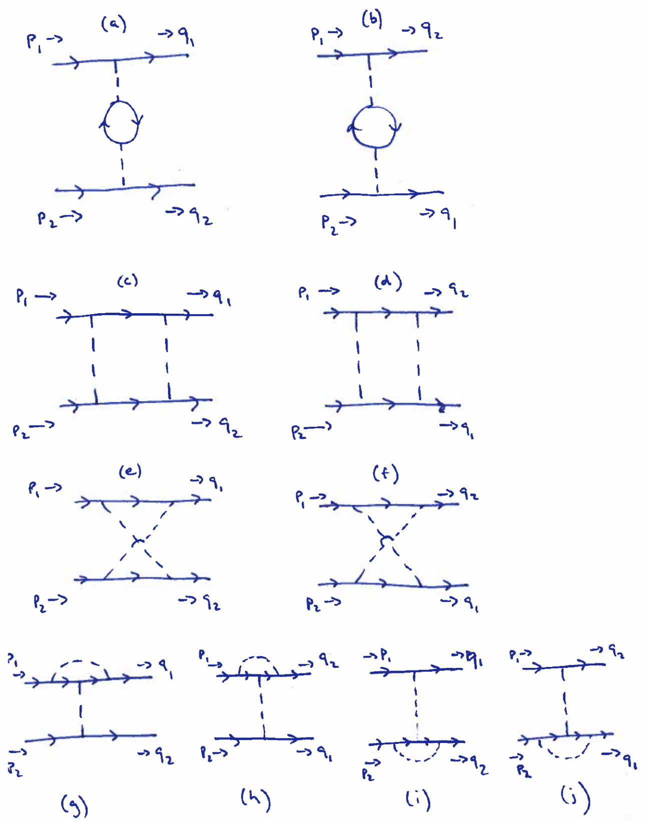
$$
\begin{aligned}
I^{(c)}_{g^4} &= g^4\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+m^2+i\epsilon)((k+p_1)^2+M^2+i\epsilon)}\\
&\quad\times\frac{1}{((k+p_1-q_1)^2+m^2+i\epsilon)((k-p_2)^2+M^2+i\epsilon)}\\
&\quad\times(2\pi)^4\delta^{(4)}(q_1-p_1+q_2-p_2),
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
I^{(e)}_{g^4} &= g^4\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+m^2+i\epsilon)((k+p_1)^2+M^2+i\epsilon)}\\
&\quad\times\frac{1}{(k+p_1-q_1)^2+m^2+i\epsilon}\\
&\quad\times\frac{1}{(k+p_1+p_2-q_1)^2+M^2+i\epsilon},\\
&\quad\times(2\pi)^4\delta^{(4)}(q_1-p_1+q_2-p_2),
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
I^{(a)}_{g^4} &= g^4\frac{1}{(p_2+q_2)^2+m^2+i\epsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+m^2+i\epsilon)}\\
&\quad\times\frac{1}{(k+p_1+p_2+q_2)^2+M^2+i\epsilon}\\
&\quad\times\frac{1}{(k+q_1)^2+M^2+i\epsilon}\\
&\quad\times(2\pi)^4\delta^{(4)}(q_1-p_1+q_2-p_2).
\end{aligned}
\tag{41}
$$

Diagrams (b), (d), and (f) can be obtained from diagrams (a), (c), and (e), respectively, by replacing q_1 with

FIG. 4. Feynman diagrams that contribute to nucleon-nucleon scattering at $\mathcal{O}(g^4)$ in perturbation theory. The solid lines represent the complex scalar field propagator D_F^ψ and the dashed lines the real scalar field propagator D_F^ϕ. Each interaction vertex is associated with a factor of the coupling constant $-ig$ in momentum space.



q_2. Similarly, diagrams (h), (i), and (j) can be obtained by suitable replacement of the external and internal momenta. Up to multiplicative prefactors, which account for the relevant symmetry factors, the sum of these contributions gives the scattering amplitude at next-to-next-to-leading order, T_{g^4}.

IV. CONCLUSION

The scalar Yukawa model, which includes one complex and one real scalar field, is a scalar model of nucleon-meson interactions. In this mini-project, I studied the scalar analogue of nucleon-nucleon scattering at $\mathcal{O}(g^4)$ in perturbation theory. Using Wick's theorem I expressed the scattering amplitude in terms of Wick contracted pairs of fields and compared that to corresponding expressions obtained by inspection of Feynman diagrams.

The scalar Yukawa model is conceptually relatively straightforward, but has several serious shortcomings as a theory of meson-nucleon interactions: most importantly, nucleons are not scalars, they are fermions. Extending this analysis to the more realistic Yukawa theory, which couples fermion fields to a scalar mediator (i.e. the pion), is relatively straightforward. Fermionic fields, however, obey anticommutation relations and this complicates the analysis of the scattering amplitude using Wick's theorem, as the new minus signs must be taken into account when normal ordering fermion fields. The propagator for the nucleon will now be the Green function of the Dirac operator, rather than the Klein-Gordon operator, and this alters the corresponding Feynman rule. The Feynman diagrams will be unchanged, although the explicit expressions for each diagram, and their relative weights, will be altered.

A second shortcoming, of both the scalar and fermionic Yukawa theories, is that physical mesons and nucleons are bound states of QCD. Although this can be partially captured through the use of form factors in the meson-nucleon coupling, this modification has limited application and the results should only be considered valid in low energy regimes. At higher energies, the partonic constituents of the mesons and nucleons are probed and a full description of nucleon-nucleon scattering would require QCD.

Finally, the scalar Yukawa theory is unbounded from below for sufficiently large coupling constant. This leads to inconsistencies in the theory as it becomes energetically favourable to create infinitely many particles. This issue, however, can be circumvented by restricting our attention to the perturbative regime, which may limit the applicability of the results.

Extending the analysis, either in the scalar or fermionic versions of the model, beyond $\mathcal{O}(g^4)$ would benefit from automated approaches to perturbation theory, because the number of diagrams and the complexity of the resulting momentum integrals increases rapidly with the order in perturbation theory. This work is underway.

Appendix A: Grading and rubric

This homework is graded out of forty points. There are ten points available for clarity of presentation. The abstract, introduction, and conclusion are worth two, four, and four points respectively, for a total of ten. These points are awarded for the content of these sections, rather than the presentation. So typographic errors or poor grammar in these sections would be accounted for in the "clarity of presentation points". For full points, the conclusion should include mention of the fact that the physical nucleons are fermions, plus at least one other salient point (including, but not limited to: composite nature of mesons and nucleons; alteration of Wick's theorem for fermions; need for automation at higher orders in perturbation theory; the use of QCD rather than chiral models for realistic scattering results; different kinematic limits; and the unboundedness of scalar Yukawa theory). The remaining twenty points are for the calculations.

Appendix B: Hints and tips

1. Clear communication

1. Try to avoid long words when short ones will do.

2. The same applies to sentences.

3. Your job is to make your reader's life easier, not yours. Try to avoid abbreviations and symbols wherever possible, unless they are extremely common and very tedious to write out for everyone involved ("QCD" is an example in my own field).

4. Think of papers that you have read that you thought were well written. Try to identify which aspects of the writing or presentation appealed to you. Try to implement or emulate those aspects wherever you can.

5. Practise. Just like solving problems helps you become a better physicist, practising communication skills helps you become a better physicist. And communication skills will serve you well, wherever life takes you[1].

2. Presentation

1. A picture is worth a thousand words. Use diagrams where possible. For example, draw the vertex, rather than describe it in words.

2. All figures should have captions describing the figures, or referring to the relevant section of the text that describes the figures (if the description is too long for the caption).

3. Communicating physics

1. Try to motivate your work through physics. This is harder than it seems. One approach to this is to think about someone who isn't interested in your field-what hook could you use to interest them? One of the keys is drawing broad connections and illustrating how your (simple toy model calculation) has bearing (however remote) with something in their arena of interest. But be aware that promising too much is nearly as bad as promising too little. Be realistic.

4. Communicating mathematics

1. You should define every symbol you use. You cannot assume that the reader is familiar with your notation (or the notation of the textbook you use). For example, if you write $D_F(x-y)$ as your Feynman rule, you should specify what this means.

2. Words can often be worth several mathematical symbols. Write out simple expressions rather than using a symbol, where possible and reasonable. For example, write "for all natural numbers", rather than "$\forall i \in \mathbb{N}$".

3. Try to avoid, as much as possible, starting sentences with mathematical symbols. It is much harder to parse "Here is a bunch of stuff and things. α was the beginning. Ω is the end." than "Here is a bunch of stuff and things. Recall, α was the beginning and Ω is the end." This is particularly true for a sentence starting directly after an equation.

It is simpler to represent each element of the Feynman rules graphically (as Peskin and Schroeder do) than to write out the description (as Tong generally does).

[1] I literally cannot think of a single plausible situation in which being a worse communicator would benefit you.

[1] Peskin, M. and Shroeder, D., *An Introduction to Quantum Field Theory*, CRC Press, 2019.

[2] Tong, D., *Quantum Field Theory*, Part III Maths Tripos Lecture Notes.