Quantum Field Theory I: PHYS 721
Problem Set 4: Solutions

Chris Monahan

Overview

The questions in this problem set first reinforce the close relationship between symmetries and conservation laws, and then familiarise you with deriving Feynman rules for a new example of an interacting theory.

Question 1 [7]

(a) Use the expressions for the Hamiltonian and the momentum operator

$$H_0 = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} E_k a^\dagger(\vec{k})a(\vec{k}),$$
$$P^i = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} k^i a^\dagger(\vec{k})a(\vec{k})$$

for a free, real scalar field, $\phi(x)$, to show that the four-vector (H, P^i) generates spacetime translations

$$\phi(x) = e^{i(Ht-\vec{P}\cdot\vec{x})}\phi(0)e^{-i(Ht-\vec{P}\cdot\vec{x})}.$$

This is another example of the phenomenon we have seen previously: the conserved charge due to a symmetry generates the corresponding symmetry transformation on the fields.

[Hint: You will need to consider expressions of the form $e^{iH_0t}a(\vec{k})e^{-iH_0t}$.]

(b) Use this result to show that

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi(x-y)\phi(0)|0>.$$

Solution 1

(a) Following the hint given in the question, we first study

$$e^{iH_0t}a(\vec{k})e^{-iH_0t} = \sum_{n=0}^{\infty}\frac{(iH_0t)^n}{n!}a(\vec{k})e^{-iH_0t}.$$

Then, using

$$[H_0, a(\vec{k})] = -E_k a(\vec{k}),$$

we have

$$H_0 a(\vec{k}) = a(\vec{k})(H_0 - E_k).$$

This means that

$$H_0^n a(\vec{k}) = a(\vec{k})(H_0 - E_k)^n.$$

Therefore we find

$$\begin{aligned} e^{iH_0t} a(\vec{k}) e^{-iH_0t} &= a(\vec{k}) \sum_{n=0}^{\infty} \frac{(iH_0t - iE_kt)^n}{n!} e^{-iH_0t} \\ &= a(\vec{k}) e^{iH_0t - iE_kt} e^{-iH_0t} \\ &= a(\vec{k}) e^{-iE_kt}. \end{aligned}$$

Similar arguments lead us to

$$\begin{aligned} e^{iH_0t} a^\dagger(\vec{k}) e^{-iH_0t} &= a^\dagger(\vec{k}) e^{iE_kt}, \\ e^{-i\vec{P}\cdot\vec{x}} a(\vec{k}) e^{i\vec{P}\cdot\vec{x}} &= a(\vec{k}) e^{i\vec{k}\cdot\vec{x}}, \\ e^{-i\vec{P}\cdot\vec{x}} a^\dagger(\vec{k}) e^{i\vec{P}\cdot\vec{x}} &= a^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}. \end{aligned}$$

For the last two equations we have used

$$\begin{aligned} [P^i, a(\vec{k})] &= \, - k^i a(\vec{k}), \\ [P^i, a^\dagger(\vec{k})] &= k^i a(\vec{k}). \end{aligned}$$

Now let's take the right hand side of the equation we wish to prove. This is

$$e^{i(Ht - \vec{P}\cdot\vec{x})} \phi(0) e^{-i(Ht - \vec{P}\cdot\vec{x})} = e^{i(Ht - \vec{P}\cdot\vec{x})} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \Big(a(\vec{k}) + a^\dagger(\vec{k}) \Big) e^{-i(Ht - \vec{P}\cdot\vec{x})}.$$

But

$$e^{i(Ht - \vec{P}\cdot\vec{x})} a(\vec{k}) e^{-i(Ht - \vec{P}\cdot\vec{x})} = a(\vec{k}) e^{-i(E_kt - \vec{k}\cdot\vec{x})}$$

and

$$e^{i(Ht - \vec{P}\cdot\vec{x})} a^\dagger(\vec{k}) e^{-i(Ht - \vec{P}\cdot\vec{x})} = a^\dagger(\vec{k}) e^{i(E_kt - \vec{k}\cdot\vec{x})},$$

so we have

$$\begin{aligned} e^{i(Ht - \vec{P}\cdot\vec{x})} \phi(0) e^{-i(Ht - \vec{P}\cdot\vec{x})} &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \Big(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \Big) \\ &= \phi(x). \end{aligned}$$

(b) First we need to note that $P^\mu|0\rangle = 0$ (where $P^\mu = (H, \vec{P})$, and that momentum operators commute $[P^\mu, P^\nu] = 0$, so that

$$e^{iP\cdot x}|0\rangle = |0\rangle.$$

Then we write

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}e^{iP\cdot y}\phi(0)e^{-iP\cdot y}|0\rangle \\
&= \langle 0|e^{-iP\cdot y}e^{iP\cdot x}\phi(0)e^{-iP\cdot x}e^{iP\cdot y}\phi(0)|0\rangle \\
&= \langle 0|e^{iP\cdot(x-y)}\phi(0)e^{-iP\cdot(x-y)}\phi(0)|0\rangle \\
&= \langle 0|\phi(x-y)\phi(0)|0\rangle.
\end{aligned}$$

Question 2 **[13]**

In our mini-lectures we derived the Feynman rules for $\lambda\phi^4$ theory, which ended up being pretty simple. In fact, it is usually possible to just read off the Feynman rules directly from the Lagrangian.

Consider the following Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3,$$

which defines the so-called $\lambda\phi^3$ theory (very imaginatively named, as you can see).

(a) What is the mass dimension of the coupling λ in four spacetime dimensions? Is this coupling relevant, marginal, or irrelevant in four spacetime dimensions? In how many spacetime dimensions is λ dimensionless?

(b) Write out what you expect the relevant position-space Feynman rules to be. You do not need to provide a detailed derivation via Wick's theorem.

(c) Draw all the connected diagrams contributing to the following:

(i) The two-point function

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle$$

up to order λ^4.

(ii) The three-point function

$$\langle 0|T\{\phi(x)\phi(y)\phi(z)\}|0\rangle$$

up to order λ^2.

(iii) The four-point function

$$\langle 0|T\{\phi(w)\phi(x)\phi(y)\phi(z)\}|0\rangle$$

up to order λ^2.

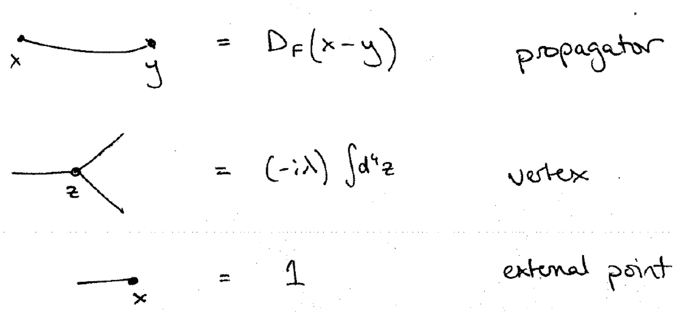

Figure 1: Feynman diagrams for the $\lambda\phi^3$ theory.

Solution 2

(a) We know that the Lagrangian density must have dimension four, because the four-integral of the Lagrangian density gives the action, which has units equal to $\hbar = 1$. Therefore we deduce that $[\phi] = 1$, so $[\phi^3] = 3$. This means that for $[\lambda\phi^3] = 4$, we must have $[\lambda] = 1$. Therefore this coupling is relevant in four spacetime dimensions.

In an arbitrary spacetime dimension d, we have $[\phi] = (d-2)/2$ (because $[m^2\phi^2] = d$), so $[\phi^3] = 3(d-2)/2$. We want to find d such that $[\lambda] = 0$. Therefore, we need to solve

$$0 + \frac{3(d-2)}{2} = d,$$

or $d = 6$. The coupling is dimensionless in six spacetime dimensions.

Note that the d-dimensional generalisation of the expression I used in class, λ/E^{4-n} is not λ/E^{d-n}. It is in fact $\lambda/E^{d-n/2(d-2)}$. This follows from the fact that the field dimension is $[\phi] = (d-2)/2$. One cannot assume $[\phi] = 1$, since this holds in only four dimensions.

(b) The position space Feynman rules are shown in Fig. 1.

(c) First note that disconnected diagrams do contribute to $\langle 0|\mathcal{O}[\{\phi_n\}]|0\rangle$, which is in contrast to $\langle \Omega|\mathcal{O}[\{\phi_n\}]|\Omega\rangle$! I show disconnected and connected diagrams for completeness, but only connected are required by the question. This significantly reduces the number of diagrams.

(i) The diagrams contributing to the two-point function are shown in Fig. 2.

(ii) The diagrams contributing to the three-point function are shown in Fig. 3.

(iii) The diagrams contributing to the four-point function are shown in Fig. 4.

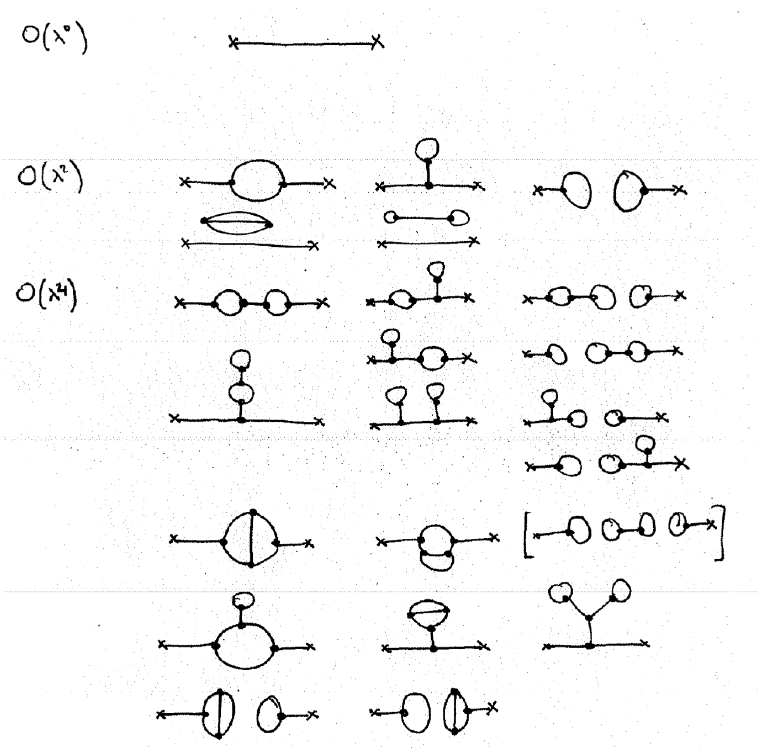

Figure 2: Connected and disconnected diagrams contributing to the two-point function in the $\lambda\phi^3$ theory.

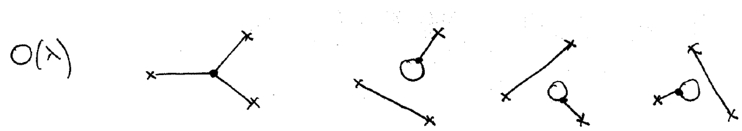

Figure 3: Connected and disconnected diagrams contributing to the three-point function in the $\lambda\phi^3$ theory.

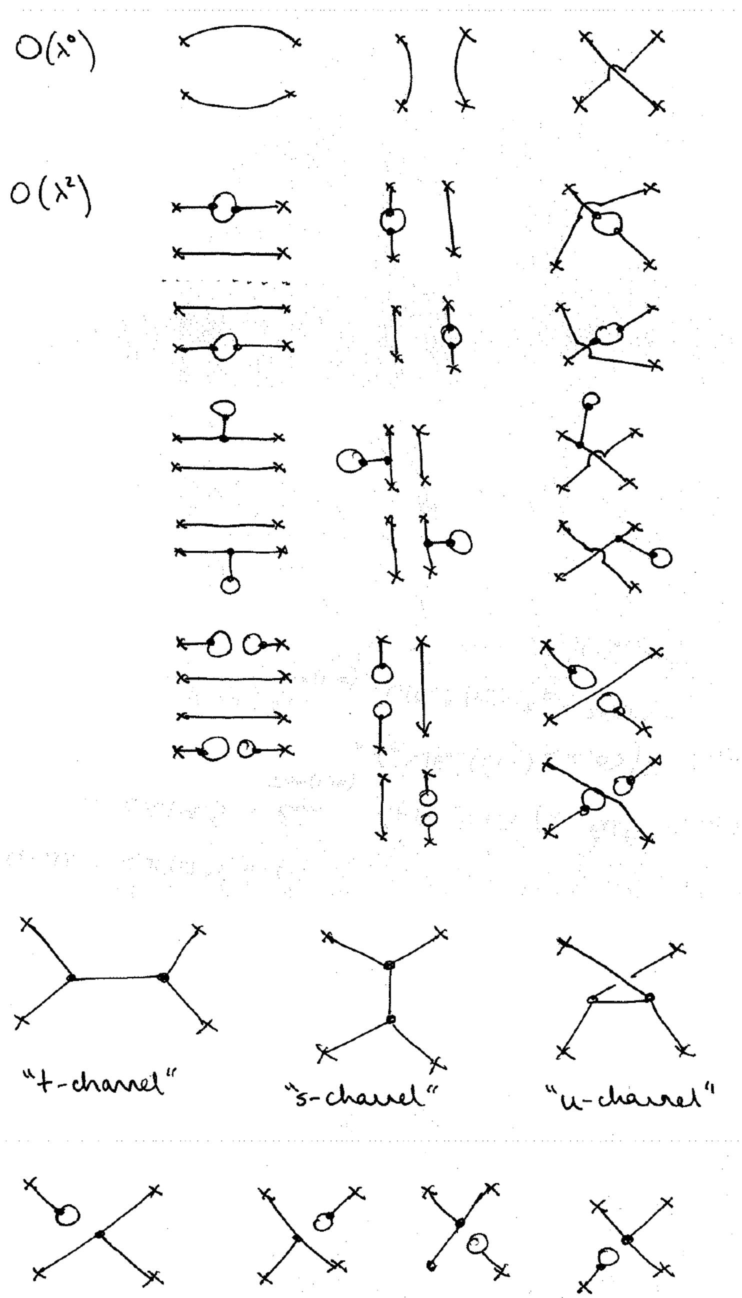

Figure 4: Connected and disconnected diagrams contributing to the four-point function in the $\lambda\phi^3$ theory.