Quantum Field Theory I: PHYS 721
Problem Set 3: Solutions

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Overview

The questions in this problem set examine the role of symmetries in (free) scalar field theories, and reinforce our understanding that the field in the interaction picture behaves like the free scalar field we studied in the first part of the course.

Question 1 **[13]**

Recall the complex scalar field that we studied in Problem Set 2, defined by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{1}$$

(a) Show that the Lagrangian density of this theory is invariant under the transformations

$$\phi(x) \to e^{i\alpha}\phi(x), \qquad \phi^*(x) \to e^{-i\alpha}\phi^*, \tag{2}$$

where α is a real-valued constant. This symmetry is, in fact, the symmetry that generates the conserved charge that we examined in Homework 2,

$$Q = \frac{i}{2}\int \mathrm{d}^3\vec{x}\,(\phi^*\pi^* - \pi\phi)\,. \tag{3}$$

(b) Show that the Lagrangian density is also invariant under *charge-conjugation*, which is defined to act on the fields as

$$\mathcal{C}\phi(x)\mathcal{C}^{-1} = \eta_C \phi^*(x). \tag{4}$$

Here $\mathcal{C}$ is a unitary operator that leaves the free-field vacuum invariant, $\mathcal{C}|0\rangle = |0\rangle$, and η_C is an arbitrary phase with unit normalisation, $|\eta_C|^2 = 1$.

(c) Show that

$$\mathcal{C}a(\vec{p})\mathcal{C}^{-1} = \eta_C b(\vec{p}), \qquad \text{and} \qquad \mathcal{C}b(\vec{p})\mathcal{C}^{-1} = \eta_C^* a(\vec{p}). \tag{5}$$

Use these results to explain how we interpret the effect of charge conjugation on particles and antiparticles. Is there a conserved quantity associated with the invariance of the Lagrange density under charge conjugation (explain your answer!)?

(d) Consider now the case of two complex fields, both with the same mass. Denote these fields by ϕ_a, where $a \in \{1, 2\}$. Use the definition of a conserved charge, Q, in terms of a conserved current, j^μ,

$$Q = \int \mathrm{d}^3\vec{x}\, j^0(\vec{x}, t), \qquad \text{where} \qquad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a \tag{6}$$

to show that there are four conserved currents, given by

$$Q = \frac{i}{2}\int \mathrm{d}^3\vec{x}\,(\phi_a^* \pi_a^* - \pi_a \phi_a)\,, \tag{7}$$

$$Q^i = \frac{i}{2}\int \mathrm{d}^3\vec{x}\,\Big(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b\Big)\,. \tag{8}$$

Here $i \in \{1, 2, 3\}$ and the σ^i are the 2×2 Pauli sigma matrices. These matrices are the generators of $SU(2)$, which is the symmetry group of angular momentum and particle spin (in other words, these Pauli sigma matrices have the same commutation relations as angular momentum operators). Note that the overall sign and constant are chosen to match the single field case.

[Hint: you will first need to think about what transformation leaves this Lagrangian density invariant.]

Solution 1

(a) We apply the transformation to the fields and define

$$\Phi^{(*)} = e^{(-)i\alpha}\phi^{(*)}, \qquad \text{and} \qquad \mathcal{L}' = \partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi.$$

Then

$$\begin{aligned}
\mathcal{L}' &= \partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi \\
&= \partial_\mu e^{-i\alpha}\phi^*\partial^\mu e^{i\alpha}\phi - m^2 e^{-i\alpha}\phi^* e^{i\alpha}\phi \\
&= e^{-i\alpha}e^{i\alpha}\Big(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\Big) \\
&= \mathcal{L}.
\end{aligned}$$

(b) Now define $\mathcal{L}'$ as the charge-conjugated Lagrangian

$$\begin{aligned}
\mathcal{L}' &= \mathcal{C}\mathcal{L}\mathcal{C}^{-1} \\
&= \partial_\mu\mathcal{C}\phi^*\mathcal{C}^{-1}\mathcal{C}\partial^\mu\phi\mathcal{C}^{-1} - m^2\mathcal{C}\phi^*\mathcal{C}^{-1}\mathcal{C}\phi\mathcal{C}^{-1} \\
&= \partial_\mu\eta_C^*\phi\partial^\mu\eta_C\phi^* - m^2\eta_C^*\phi\eta_C\phi^* \\
&= \eta_C^*\eta_C\Big(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\Big) \\
&= \mathcal{L}.
\end{aligned}$$

Here we've used $|\eta_C|^2 = 1$.

(c) We first write the creation and annihilation operators in terms of the fields, as

$$
\begin{aligned}
a(\vec{p}) &= -\frac{i}{(2\pi)^3\sqrt{E_p}}\int \mathrm{d}^3\vec{x}\Big(iE_p\phi(\vec{x}) - \partial_0\phi(\vec{x})\Big)e^{-i\vec{p}\cdot\vec{x}} \\
b(\vec{p}) &= -\frac{i}{(2\pi)^3\sqrt{E_p}}\int \mathrm{d}^3\vec{x}\Big(iE_p\phi^*(\vec{x}) - \partial_0\phi^*(\vec{x})\Big)e^{-i\vec{p}\cdot\vec{x}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{C}a(\vec{p})\mathcal{C}^{-1} &= -\frac{i}{(2\pi)^3\sqrt{E_p}}\int \mathrm{d}^3\vec{x}\Big(iE_p\mathcal{C}\phi(\vec{x})\mathcal{C}^{-1} - \partial_0\mathcal{C}\phi(\vec{x})\mathcal{C}^{-1}\Big)e^{-i\vec{p}\cdot\vec{x}} && (9)\\
&= -\frac{i}{(2\pi)^3\sqrt{E_p}}\int \mathrm{d}^3\vec{x}\Big(iE_p\eta_C\phi^*(\vec{x}) - \partial_0\eta_C\phi^*(\vec{x})\Big)e^{-i\vec{p}\cdot\vec{x}} && (10)\\
&= \eta_C b(\vec{p}), && (11)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{C}b(\vec{p})\mathcal{C}^{-1} &= -\frac{i}{(2\pi)^3\sqrt{E_p}}\int \mathrm{d}^3\vec{x}\Big(iE_p\mathcal{C}\phi^*(\vec{x})\mathcal{C}^{-1} - \partial_0\mathcal{C}\phi^*(\vec{x})\mathcal{C}^{-1}\Big)e^{-i\vec{p}\cdot\vec{x}} \\
&= -\frac{i}{(2\pi)^3\sqrt{E_p}}\int \mathrm{d}^3\vec{x}\Big(iE_p\eta_C^*\phi(\vec{x}) - \partial_0\eta_C^*\phi(\vec{x})\Big)e^{-i\vec{p}\cdot\vec{x}} \\
&= \eta_C^* a(\vec{p}).
\end{aligned}
$$

Then by complex conjugation of these results, we obtain

$$
\begin{aligned}
\mathcal{C}a^\dagger(\vec{p})\mathcal{C}^{-1} &= \eta_C^* b^\dagger(\vec{p}), \\
\mathcal{C}b^\dagger(\vec{p})\mathcal{C}^{-1} &= \eta_C a^\dagger(\vec{p}).
\end{aligned}
$$

If we denote a state of charge q and momentum $\vec{p}$ as $|q;\vec{p}\rangle$, then we have

$$
\mathcal{C}|q;\vec{p}\rangle = \mathcal{C}a(\vec{p})|0\rangle = \mathcal{C}a(\vec{p})\mathcal{C}^{-1}\mathcal{C}|0\rangle = \eta_C^* b^\dagger(\vec{p})|0\rangle = \eta_C^*|-q,\vec{p}\rangle.
$$

Thus charge conjugation turns a particle into an antiparticle. Similar arguments show that charge conjugation also turns antiparticles into a particle.

There is no conserved current associated with charge conjugation, because it is a discrete symmetry and Noether's theorem applies to continuous symmetries.

(d) For the case of two complex fields, the Lagrangian density is given by

$$
\mathcal{L} = \partial_\mu\phi_a^*\partial^\mu\phi^a - m^2\phi_a^\dagger\phi_a,
$$

where $a \in \{1, 2\}$ labels the "species" of particle.

The Lagrangian density for the single particle case was invariant under a complex phase rotation, which is a $U(1)$ symmetry. With two particles, the Lagrangian density is symmetric under $U(2)$ symmetry, $\phi_a \to U_{ab}\phi_b$. The $U(2)$ group is of dimension four, so is spanned by four generators, which can be chosen to be the $SU(2)$ Pauli matrices and the identity. So, for example,

$$\phi_a \to U_{ab}\phi_b = e^{-i\tau_{ab}/2}\phi_b,$$

so that

$$\delta\phi_a = -\frac{i}{2}\tau_{ab}\phi_b.$$

Note that the Pauli matrices are Hermitian.

Then the conserved currents are given by

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a^*)}\delta\phi_a^* \\ &= \partial_\mu\phi_a^*(-i\phi_a) + \partial_\mu\phi_a(i\phi_a^*) \end{aligned}$$

for transformation under the identity, and

$$\begin{aligned} j^{\mu,i} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta^i\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a^*)}\delta^i\phi_a^* \\ &= \partial_\mu\phi_a^*\left(-\frac{i}{2}\sigma^i_{ab}\phi_b\right) + \partial_\mu\phi_a\left(\frac{i}{2}\sigma^i_{ab}\phi_b^*\right). \end{aligned}$$

Then the corresponding conserved currents are

$$\begin{aligned} Q &= \int \mathrm{d}^3\vec{x}\, j^0 \\ &= i\int \mathrm{d}^3\vec{x}\,\left(\dot{\phi}_a\phi_a^* - \dot{\phi}_a^*\phi_a\right) \\ &= i\int \mathrm{d}^3\vec{x}\,\left(\pi_a^*\phi_a^* - \pi_a\phi_a\right) \end{aligned}$$

and

$$Q^i = \int \mathrm{d}^3\vec{x}\, j^{0,i} \tag{12}$$

$$= \frac{i}{2}\int \mathrm{d}^3\vec{x}\,\left(\dot{\phi}_a\sigma^i_{ab}\phi_b^* - \dot{\phi}_a^*\sigma^i_{ab}\phi_b\right) \tag{13}$$

$$= \frac{i}{2}\int \mathrm{d}^3\vec{x}\,\left(\pi_a^*\sigma^i_{ab}\phi_b^* - \pi_a\sigma^i_{ab}\phi_b\right). \tag{14}$$

Under the assumption that these operators are normal ordered, we can write them as

$$Q = \frac{i}{2}\int \mathrm{d}^3\vec{x}\,(\phi_a^*\pi_a^* - \pi_a\phi_a)\,, \tag{15}$$

$$Q^i = \frac{i}{2}\int \mathrm{d}^3\vec{x}\left(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b\right). \tag{16}$$

Question 2 **[7]**

In lectures we claimed that the scalar field in the interaction picture, which is defined as $\phi(\vec{x},t) = e^{iH_0t}\phi(\vec{x},0)e^{-iH_0t}$, can be decomposed in terms of Fourier modes in exactly the same way as the free-field (which satisfies the Klein-Gordon equation). Show that ϕ_I does indeed satisfy the Klein-Gordon equation, thus justifying this claim.

[Hint: You will probably want to calculate the commutators $[H_0, \phi(\vec{x},0)]$ and $[H_0, \pi(\vec{x},0)]$ to help you simplify things.]

Solution 2

The interaction picture field is given by

$$\phi_I(\vec{x},t) = e^{iH_0t}\phi(\vec{x},0)e^{-iH_0t}.$$

We need to find the second derivative (with respect to time) of this field, so we need

$$\frac{\partial}{\partial t}\phi_I(\vec{x},t) = iH_0\phi_I(\vec{x},t) - i\phi_I(\vec{x},t)H_0,$$

and then

$$\begin{aligned}\frac{\partial}{\partial t}\phi_I(\vec{x},t) &= iH_0\left(iH_0\phi_I(\vec{x},t) - i\phi_I(\vec{x},t)H_0\right) - i\left(iH_0\phi_I(\vec{x},t) - i\phi_I(\vec{x},t)H_0\right)H_0\\ &= -H_0^2\phi_I(\vec{x,t}) + 2H_0\phi_I(\vec{x,t})H_0 - \phi_I(\vec{x,t})H_0^2\\ &= e^{iH_0t}\left(-H_0^2\phi(\vec{x,0}) + 2H_0\phi(\vec{x,0})H_0 - \phi(\vec{x,0})H_0^2\right)e^{-iH_0t}.\end{aligned}$$

Now use the commutators suggested in the question to simplify this. First note that

$$H_0 = \int \mathrm{d}^3\vec{x}\left(\pi^2(\vec{x},0) + \phi(\vec{x},0)(-\vec{\delta}^2 + m^2)\phi(\vec{x},0)\right).$$

Then we have (recall Problem Set 2)

$$[H_0, \phi(\vec{x}, 0)] = -i\pi(\vec{x}, 0),$$
$$[H_0, \pi(\vec{x}, 0)] = i(-\vec{\nabla}^2 + m^2)\phi(\vec{x}, 0).$$

From these, we can deduce that

$$\begin{aligned} H_0^2\phi &= H_0(H_0\phi) \\ &= H_0([H_0, \phi] + \phi H_0) \\ &= H_0(-i\pi + \phi H_0) \\ &= -iH_0\pi + H_0\phi H_0 \end{aligned}$$

and

$$\begin{aligned} \phi H_0^2 &= (\phi H_0)H_0 \\ &= ([\phi, H_0] + H_0\phi)H_0 \\ &= (i\pi + H_0\phi)H_0 \\ &= i\pi H_0 + H_0\phi H_0. \end{aligned}$$

Then we have

$$\begin{aligned} I &\equiv -H_0^2\phi + 2H_0\phi H_0 - \phi H_0^2 \\ &= -(-iH_0\pi + H_0\phi H_0) + 2H_0\phi H_0 - (i\pi H_0 + H_0\phi H_0) \\ &= i[H_0, \pi] \\ &= (\vec{\nabla}^2 - m^2)\phi(\vec{x}, 0). \end{aligned}$$

Putting this all together, we have

$$\begin{aligned} \frac{\partial}{\partial t}\phi_I(\vec{x}, t) &= e^{iH_0t}\left(-H_0^2\phi(\vec{x,0}) + 2H_0\phi(\vec{x,0})H_0 - \phi(\vec{x,0})H_0^2\right)e^{-iH_0t} \\ &= e^{iH_0t}\left((\vec{\nabla}^2 - m^2)\phi(\vec{x}, 0)\right)e^{-iH_0t} \\ &= (\vec{\nabla}^2 - m^2)\phi_I(\vec{x}, t). \end{aligned}$$

Rearranging this gives the Klein-Gordon equation.