Quantum Field Theory I: PHYS 721
Problem Set 9

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The questions in this problem set will explore some properties of QED.

Question 1 [8]

Gauge invariance is a strong constraint in field theories, and manifests in many different ways. One manifestation of gauge invariance is the existence of Ward identities, which constrain the form of scattering amplitudes.

In general, if a scattering amplitude involves external photons (let's say there are i incoming external photons and j outgoing), so that the invariant matrix element can be written as

$$\mathcal{M}(A+\sum_i \gamma_i \to B+\sum_j \gamma_j) = \epsilon^{r_1}_{\mu_1}(k_1)\cdots\epsilon^{r_i}_{\mu_i}(k_i)\epsilon^{r_{i+1}*}_{\mu_{i+1}}(k_{i+1})\cdots\epsilon^{r_n*}_{\mu_n}(k_n)\mathcal{M}^{\mu_1\cdots\mu_n}(k_1,\cdots,k_n),$$

where $n = i + j$, then the Ward identity implies

$$k^{\mu_a}\mathcal{M}_{\mu_1\cdots\mu_n}(k_1,\cdots,k_n) = 0, \tag{1}$$

for any $a \in \{1, n\}$.

(a) Consider the case of Compton scattering. The leading order contributions are given by

$$i\mathcal{M}^{(1)} = (-ig)^2\epsilon^*_\mu(k_2)\epsilon_\nu(k_1)\overline{u}(p_2)\gamma^\mu\frac{i(\not{p}_1 + \not{k}_1 + m)}{(p_1 + k_1)^2 - m^2}\gamma^\nu u(p_1),$$
$$i\mathcal{M}^{(2)} = (-ig)^2\epsilon^*_\mu(k_2)\epsilon_\nu(k_1)\overline{u}(p_2)\gamma^\nu\frac{i(\not{p}_1 - \not{k}_2 + m)}{(p_1 - k_2)^2 - m^2}\gamma^\mu u(p_1).$$

Show that individually the terms do not satisfy the Ward identity,

$$k_2^\mu\mathcal{M}^{(1)}_{\mu\nu} \neq 0 \qquad \text{and} \qquad k_2^\mu\mathcal{M}^{(2)}_{\mu\nu} \neq 0,$$

but the sum does:

$$k_2^\mu\left(\mathcal{M}^{(1)}_{\mu\nu} + \mathcal{M}^{(2)}_{\mu\nu}\right) = 0.$$

Note, you should not need more than about a page of algebra! If you find yourself three pages in, you have probably not chosen the most efficient way to do this!

Question 2

(a) Evaluate the one-fermion-loop contribution to the photon one-point function, $\langle\Omega|A_\mu|\Omega\rangle$, and show that it vanishes. You shouldn't need to explicitly carry out the integral to show that the result is zero.

(b) Now consider the photon three-point function, $\langle\Omega|A_\mu A_\nu A_\rho|\Omega\rangle$, and write down the two diagrams that contribute to this vacuum expectation value. Show that individually these contributions is nonzero, but that their sum is zero.

(c) By considering the properties under charge transformation of photon n-point functions, show that these functions vanish if n is odd. Note that this result is true nonperturbatively and you should not need to assume a perturbative expansion.