Quantum Field Theory I: PHYS 721
Problem Set 8

Chris Monahan

The question in this problem set will explore the effects of an external magnetic field on the relativistic energy levels in hydrogen.

Question

Consider an electron in a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, with $B > 0$ and $\hat{\mathbf{z}}$ a unit vector in the z-direction. The gauge potential is defined through $\mathbf{B} = \nabla \times \mathbf{A}$, so we can take $A^\mu(x^\mu) = (0, 0, Bx, 0)$, where $x^\mu = (0, \mathbf{x}) = (0, x, 0, 0)$.

(a) Write the fermion field in terms of two two-component spinors, $\phi(x^\mu)$ and $\chi(x^\mu)$, as

$$\Psi(x^\mu) = \begin{pmatrix} \phi(x^\mu) \\ \chi(x^\mu) \end{pmatrix}$$

and use the principle of minimal coupling to show that these two-spinors satisfy the Dirac equations,

$$(i\partial_0 - m)\phi(x^\mu) = \sigma \cdot (\mathbf{p} - e\mathbf{A})\chi(x^\mu) \qquad \text{, and} \qquad (i\partial_0 + m)\chi(x^\mu) = \sigma \cdot (\mathbf{p} - e\mathbf{A})\phi(x^\mu),$$

where $\mathbf{p} = -i\nabla$ and we assume the Dirac basis for the gamma matrices.

(b) Assume plane-wave solutions of the form

$$\phi(x) = \phi(\mathbf{x})e^{-iEt} \qquad \text{and} \qquad \chi(x) = \chi(\mathbf{x})e^{-iEt}$$

to derive coupled equations for the spatial dependence of both two-spinors. Eliminate the dependence on $\chi(\mathbf{x})$ to obtain a single differential equation for $\phi(\mathbf{x})$.

(c) The components of the momentum p^y and p^z commute with x, so we can search for a solution of the form

$$\phi(\mathbf{x}) = e^{i(p^y y + p^z z)}\nu(x),$$

where $\nu(x)$ is a two-spinor. Furthermore, we can assume that the electron has spin in the z-direction, so that $\nu(x)$ is an eigenfunction of σ_z with eigenvalue $\lambda = \pm 1$. Show that $\nu(x)$ satisfies

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2} \cdot 2e^2B^2 \left(x - \frac{p^y}{eB} \right)^2 \right] \nu(x) = \left(E^2 - m^2 - (p^z)^2 + \lambda eB \right) \nu(x)$$

(d) This equation is formally identical to the Schrödinger equation for a well-studied quantum mechanical system. Identify the system and the relevant physical parameter that characterises this system. Use this to identify the energy levels of the system and show that they can be written as

$$E(n, p^z, \lambda) = \sqrt{m^2 + (p^z)^2 + (2n + 1 + \lambda)|e|B},$$

where the charge of the electron is $e = -|e|$. Identify the degeneracies in this system.

(e) Show that this result for the energy levels reduces to the usual nonrelativistic (Landau) levels for a spin-1/2 particle in an external magnetic field, in the nonrelativistic limit $(p^z)^2 \ll m^2$ and $(2n + 1)|e|B \ll m^2$.