Quantum Field Theory I: PHYS 721
Problem Set 6

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The questions in this problem give you some practice at manipulating spinors and gamma matrices.

Question 1

(a) Prove the spinor relation

$$\overline{u}^r(\vec{p})\gamma^\mu u^s(\vec{q}) = \frac{1}{2m}\overline{u}^r(\vec{p})\left[p^\mu + q^\mu + i\sigma^{\mu\nu}(p_\nu - q_\nu)\right]u^s(\vec{q}).$$

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. This relation is known as the "Gordon identity". Note that this is named after Walter Gordon of Klein-Gordon equation fame, and not Paul Gordan of Clebsch-Gordan coefficient fame.

(b) Using the identity

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta},$$

where $\epsilon_{12} = +1$, show that

$$\left[\overline{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_2\right]\left[\overline{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_4\right] = -\left[\overline{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_4\right]\left[\overline{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_2\right].$$

Here $\overline{u}_{1,3}$ and $u_{2,4}$ are four different spinors. This is an example of a "Fierz identity". These identities relate products of spinor bilinears to sums of products of more useful spinor bilinears.

Both the Gordon and various Fierz identities are often used in calculations of scattering amplitudes involving fermions.

Question 2

(a) Show that the vector representation of the Lorentz group satisfies the Lorentz Lie algebra, that is, show that

$$[\overset{1}{J}{}^{\mu\nu}, \overset{1}{J}{}^{\rho\sigma}] = i\left(g^{\nu\rho}\,\overset{1}{J}{}^{\mu\sigma} - g^{\mu\rho}\,\overset{1}{J}{}^{\nu\sigma} - g^{\nu\sigma}\,\overset{1}{J}{}^{\mu\rho} + g^{\mu\sigma}\,\overset{1}{J}{}^{\nu\rho}\right).$$

(b) Show the spinor representation of the Lorentz group satisfies the Lorentz Lie algebra, that is, show that

$$[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \overset{\frac{1}{2}}{J}{}^{\rho\sigma}] = i\left(g^{\nu\rho}\overset{\frac{1}{2}}{J}{}^{\mu\sigma} - g^{\mu\rho}\overset{\frac{1}{2}}{J}{}^{\nu\sigma} - g^{\nu\sigma}\overset{\frac{1}{2}}{J}{}^{\mu\rho} + g^{\mu\sigma}\overset{\frac{1}{2}}{J}{}^{\nu\rho}\right).$$