Quantum Field Theory I: PHYS 721
Problem Set 3

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Overview

The questions in this problem set examine the role of symmetries in (free) scalar field theories, and reinforce our understanding that the field in the interaction picture behaves like the free scalar field we studied in the first part of the course.

Question 1 [13]

Recall the complex scalar field that we studied in Problem Set 2, defined by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{1}$$

(a) Show that the Lagrangian density of this theory is invariant under the transformations

$$\phi(x) \to e^{i\alpha}\phi(x), \qquad \phi^*(x) \to e^{-i\alpha}\phi^*, \tag{2}$$

where α is a real-valued constant. This symmetry is, in fact, the symmetry that generates the conserved charge that we examined in Homework 2,

$$Q = \frac{i}{2}\int \mathrm{d}^3\vec{x}\,(\phi^*\pi^* - \pi\phi)\,. \tag{3}$$

(b) Show that the Lagrangian density is also invariant under *charge-conjugation*, which is defined to act on the fields as

$$\mathcal{C}\phi(x)\mathcal{C}^{-1} = \eta_C \phi^*(x). \tag{4}$$

Here $\mathcal{C}$ is a unitary operator that leaves the free-field vacuum invariant, $\mathcal{C}|0\rangle = |0\rangle$, and η_C is an arbitrary phase with unit normalisation, $|\eta_C|^2 = 1$.

(c) Show that

$$\mathcal{C}a(\vec{p})\mathcal{C}^{-1} = \eta_C b(\vec{p}), \qquad \text{and} \qquad \mathcal{C}b(\vec{p})\mathcal{C}^{-1} = \eta_C^* a(\vec{p}). \tag{5}$$

Use these results to explain how we interpret the effect of charge conjugation on particles and antiparticles. Is there a conserved quantity associated with the invariance of the Lagrange density under charge conjugation (explain your answer!)?

(d) Consider now the case of two complex fields, both with the same mass. Denote these fields by ϕ_a, where $a \in \{1, 2\}$. Use the definition of a conserved charge, Q, in terms of a conserved current, j^μ,

$$Q = \int \mathrm{d}^3\vec{x}\, j^0(\vec{x}, t), \qquad \text{where} \qquad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \tag{6}$$

to show that there are four conserved currents, given by

$$Q = \frac{i}{2} \int \mathrm{d}^3\vec{x}\, (\phi_a^* \pi_a^* - \pi_a \phi_a), \tag{7}$$

$$Q^i = \frac{i}{2} \int \mathrm{d}^3\vec{x} \left(\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b \right). \tag{8}$$

Here $i \in \{1, 2, 3\}$ and the σ^i are the 2×2 Pauli sigma matrices. These matrices are the generators of $SU(2)$, which is the symmetry group of angular momentum and particle spin (in other words, these Pauli sigma matrices have the same commutation relations as angular momentum operators). Note that the overall sign and constant are chosen to match the single field case.

[Hint: you will first need to think about what transformation leaves this Lagrangian density invariant.]

Question 2 **[7]**

In lectures we claimed that the scalar field in the interaction picture, which is defined as $\phi(\vec{x}, t) = e^{iH_0 t} \phi(\vec{x}, 0) e^{-iH_0 t}$, can be decomposed in terms of Fourier modes in exactly the same way as the free-field (which satisfies the Klein-Gordon equation). Show that ϕ_I does indeed satisfy the Klein-Gordon equation, thus justifying this claim.

[Hint: You will probably want to calculate the commutators $[H_0, \phi(\vec{x}, 0)]$ and $[H_0, \pi(\vec{x}, 0)]$ to help you simplify things.]