Quantum Field Theory I: PHYS 721
Problem Set 2

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Overview

The questions in this problem set will familiarise you with manipulating operators (both the field operators themselves and the ladder operators) and using commutation relations, and introduce a slightly different quantum field theory than the one we've seen so far. These manipulations form the basis of the first section of the course.

Question 1 [8]

Consider the free scalar field theory defined by the Hamiltonian

$$\widehat{H} = \int \mathrm{d}^3x \, \frac{1}{2} \left(\widehat{\pi}^2 + \widehat{\phi} \left(-\vec{\nabla}^2 + m^2 \right) \widehat{\phi} \right).$$

Calculate the equal-time commutation relations

$$\left[\widehat{\pi}^2(x), \widehat{\phi}(y) \right] \qquad \text{and} \qquad \left[\widehat{\phi}(x)(-\vec{\nabla}^2 + m^2)\widehat{\phi}(x), \widehat{\pi}(y) \right]$$

and use these to show that the (operator-valued) scalar field $\widehat{\phi}(x)$ satisfies the (operator-valued) Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 \right) \widehat{\phi}(x) = 0.$$

Question 2 [based on Peskin and Shroeder 2.2] [12]

In this question we study the quantum field theory of a complex scalar field, defined by the action

$$\widehat{S} = \int \mathrm{d}^4x \left(\partial_\mu \widehat{\phi}^* \partial^\mu \widehat{\phi} - m^2 \widehat{\phi}^* \widehat{\phi} \right). \tag{1}$$

We could choose to analyse this theory by treating the real and imaginary parts of the complex field $\widehat{\phi}$ as independent dynamical variables, but it is easier to instead choose $\widehat{\phi}$ and $\widehat{\phi}^*$ as the basic independent variables.

(a) Find the conjugate momenta of $\widehat{\phi}(x)$ and $\widehat{\phi}^*(x)$ and the corresponding canonical commutation relations.

(b) Show that the Hamiltonian is

$$\widehat{H} = \int \mathrm{d}^3\vec{x} \left(\widehat{\pi}^* \widehat{\pi} + \vec{\nabla}\widehat{\phi}^* \cdot \vec{\nabla}\widehat{\phi} + m^2 \widehat{\phi}^* \widehat{\phi} \right). \tag{2}$$

Determine the Heisenberg equation of motion for $\widehat{\phi}(x)$ and show that it leads to the Klein-Gordon equation.

(c) Write the Hamiltonian in terms of creation and annihilation operators[1] and show that the theory contains two sets of particles of mass m.

(d) Rewrite the conserved charge

$$\widehat{Q} = \frac{i}{2} \int \mathrm{d}^3\vec{x} \left(\widehat{\phi}^* \widehat{\pi}^* - \widehat{\pi}\widehat{\phi} \right) \tag{3}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

[1]Think about why the creation and annihilation operators were complex conjugates of each other in the real scalar field case. Does that have to be true in this case?