Quantum Field Theory I: PHYS 721
Final Project

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Overview

This project is due on **Tuesday November 24** at **5 pm**. Submit your manuscript as a PDF via email to me, and include your name in the filename of your submission. **Late submissions will not be accepted.**

Read the following carefully. The instructions are similar to, but not quite the same as, Problem Set 5.

You must submit your project as a PDF, using LaTeX with a style file from a peer-reviewed journal. Use a two-column format.

The *structure* of the Final Project follows that of Problem Set 5. You should include:

- Title;
- Author information;
- Abstract;
- Introduction;
- Conclusion;

as well as ensuring you have answered all the questions. Your manuscript should be between five and ten sides (in two column format), including references, although the lower end of this is preferred. You will be penalised for submissions longer than ten pages. You are free to use textbooks, but you must cite all references that you use.

This project will be graded out of 60 points. Fifteen of those points are for clarity of presentation. You should be sure to take note of the suggestions I gave following your Problem Set 5. You should use complete sentences throughout. Figures should have captions and you may draw Feynman diagrams by hand.

You may discuss, in general terms, the project with other students but please be aware that the final project must be your own work. You can, for example,

1. talk about general strategies;
2. discuss your approach to a given part of a problem;
3. explain your understanding of what you think the question is asking for; or
4. point out a useful reference that helped you solve something.

But you cannot, for example:

1. write passages for each other;
2. share your drafts with each other; or
3. proof read any part of another person's solutions.

This project will:

- introduce you to Fermi's theory of the weak interaction and the concept of effective theories;
- help you solidify your understanding of scattering in field theories involving fermions;
- provide an opportunity for applying the concepts and techniques of quantum field theory to an unfamiliar theory: the weak nuclear force;
- give you practice communicating your work to your professional peers.

Your abstract should be three to five sentences, explaining what you have done, how, and what your central result is.

Your introduction will consist of your solution to the first part of the project (i.e. part (a)).

Your conclusion should summarise what you've done, how, and what your central result was. It is standard to include a suggestion for future work in your conclusion, so you should incorporate your answer to the final part of the homework (i.e. part (g)) as part of your conclusion.

You do not need to answer questions in order, nor do you have to use the section headings that appear in this sheet, but you should ensure your answer to each question appears in your solution somewhere.

There are some very nice overviews of how to write well for a physics audience here:

- https://www.southwestern.edu/live/files/4179-guide-for-writing-in-physicspdf
- https://cdn.journals.aps.org/files/rmpguapa.pdf
- https://www.geneseo.edu/ mclean/Dept/JournalArticle.pdf

and you can find many others online. The second of these has useful hints for speakers whose first language is not English. The last reference aims to familiarise you with how to write a scientific article, but also has some tips for writing well. Perhaps the most important points are

1. Use the active voice where possible.
2. Be succinct, but do not omit relevant details.
3. Write so that the reader has an easy time, not you. For example, try not to use abbreviations if you can avoid them.
4. Be concrete where possible.
5. Try not to start a sentence with a mathematical symbol.
6. Punctuate mathematical equations as if they were phrases or other parts of a sentence.
7. Practise, practise, practise!

I give further specific hints in my suggested solution to Problem Set 5.

Final project

Motivation

In this project we will study muon decay in the Standard Model of particle physics. This decay is mediated by the weak force and, to motivate our understanding of this new theory, we will start by briefly considering a toy model.

In class I said that we did not need to consider four-fermion interactions, that is, interaction terms of the form $(\overline{\psi}\psi)^2$. This, however, applies only to renormalisable theories. These sort of interaction terms do appear in *effective field theories*, which are low-energy descriptions that use only the *relevant degrees of freedom* at a specific energy scale or range of energy scales. At low energies, we cannot tell what is happening at very high energies (or short-distances), and we parameterise our ignorance of the true nature of reality at short distances by using effective theories. These are only valid up to a certain momentum cutoff (i.e. down to a certain length scale), and they need not be renormalisable. Fermi's theory of the weak force is the classic example of this.

In Section 1 we look at a toy model of this theory to begin to understand how these ideas work, using two different model Lagrangians. In Section 2 we then consider muon decay $\mu^- \to e^- \overline{\nu}_e \nu_\mu$, introducing only the necessary ingredients of the weak force, which mediates this process. A full understanding of the weak force is not required to answer these questions and our focus is on applying our knowledge of scattering to a new theory.

Questions

Introduction

(a) Write an overview of scattering in field theory. Your summary should address the main steps that are required to go from the Lagrangian of a given theory to experimental observables. You should motivate each of these steps and highlight those aspects of different theories (for example, real scalar field theory, fermions, QED) that are similar or otherwise. This overview will form the introduction to your write up.

This introduction should balance completeness with brevity. In other words, you should include enough detail to show you understand the complete picture, but not so much detail that the reader is swamped.

Section 1

(b) Consider the Lagrangian of a free Dirac fermion Ψ, with a four-fermion interaction term

$$\mathcal{L}_T = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + G(\overline{\Psi}\Psi)^2.$$

Here G is a coupling constant.

i) How do we know that this theory is not renormalisable?

ii) Draw the leading-order diagram for fermion-antifermion scattering ($\Psi\overline{\Psi} \to \Psi\overline{\Psi}$) in this theory and write down the corresponding invariant matrix element.

(c) Consider now the Yukawa Lagrangian of a Dirac fermion Ψ, coupled to a massive scalar field, ϕ,

$$\mathcal{L}_Y = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - M^2\phi^2\right) - g\phi\overline{\Psi}\Psi.$$

i) How do we know that this theory *is* renormalisable?

ii) Draw the leading-order diagram for fermion-antifermion scattering ($\Psi\overline{\Psi} \to \Psi\overline{\Psi}$) in this theory and write down the corresponding invariant matrix element.

iii) Work in the centre-of-momentum frame, in which the sum of the incoming momentum is given by $(p_1 + p_2)^2 = 4E^2$, and show that at low energies the two amplitudes are the same, provided $G \propto g^2/M^2$. Find the constant of proportionality.

iv) Discuss your physical interpretation of this result.

Section 2

In the Standard Model of particle physics we have two types of fermions: quarks and leptons. There are three generations of each of these. The leptons are electrons, muons, and taus, and their neutrino partners ν_e, ν_μ, and ν_τ. The quark families are up and down, strange and charm, and top and bottom. Neglecting neutrino masses, which are much lighter than other scales in the Standard Model, means that only left-handed neutrinos couple to the Standard Model[1]. The other leptons, however, can be both left- or right-handed, and we denote the Weyl spinors representing the left- and right-handed electrons, for example, as

$$e_L = \frac{1-\gamma^5}{2}e = P_L e, \qquad e_R = \frac{1+\gamma^5}{2}e = P_R e,$$

[1] The full explanation of this requires a fuller understanding of the weak force [see, for example, Chapter 29 of Schwarz's "Quantum Field Theory and the Standard Model"], which is not necessary here, and you can take this as a given fact.

where e is a four-component Dirac spinor. The neutrino spinors can be treated in the same way, except that there are only left-handed neutrinos.

The weak force is mediated by the charged W_μ^+ and W_μ^- and neutral Z_μ^0 massive vector bosons. You can treat these as analogous to the photon in QED, except they have charge and mass. The charge only enters the problem through the coupling to the fermions, but the mass appears in the propagator. At leading order, only the charged vector bosons take part in the weak decay of the muon, and they have a propagator given by

$$\widetilde{D}_{\mu\nu}(q) = \frac{-i}{q^2 - m_W^2 + i\epsilon}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_W^2}\right).$$

Here m_W is the mass of the $W_\mu^\pm$ bosons, which is $m_W \sim 80.4$ GeV. If all the masses of the initial and final particles are much less than this mass, then we can approximate the propagator as

$$\widetilde{D}_{\mu\nu}(q) \simeq \frac{i g_{\mu\nu}}{m_W^2}.$$

You may assume that the neutrinos are massles, and that the masses of the electron and muon are much smaller than the mass of the W_μ^- boson,

$$\frac{m_e^2}{m_W^2} \ll 1, \qquad \frac{m_\mu^2}{m_W^2} \ll 1.$$

For muon decay, the relevant interaction between the W_μ^- bosons and the electron and muon lepton families is given by the interaction Lagrangian

$$\mathcal{L}_{\text{int.}} = \frac{g}{\sqrt{2}}\left(\bar{e}_L \gamma^\mu \nu_{e,L} + \bar{\nu}_{\mu,L}\gamma^\mu \mu_L\right) W_\mu.$$

This origin of this interaction term is analogous to the way we coupled the conserved vector current to the photon in QED. Here, instead, we couple the gauge boson (the massive charged $W_\mu^\pm$ boson) to the corresponding currents.

(d) i) Draw the leading order Feynman diagram for muon decay

$$\mu^- \to \nu_\mu \bar{\nu}_e e^-.$$

[Hint: this involves two different vertices that connect a lepton, its neutrino partner and a W boson. Be careful with which states are initial and final states.]

ii) Use the properties of the gamma matrices to show that the interaction can be written in "$V - A$" form

$$\mathcal{L}_{\text{int.}} = \frac{g}{\sqrt{2}}\left(\bar{e}\gamma^\mu P_L \nu_e + \bar{\nu}_\mu \gamma^\mu P_L \mu\right) W_\mu.$$

Note that here the spinors are now four-component Dirac spinors, and because the neutrino only occurs as a left-handed particle, its four-component spinor obeys

$$P_L \nu = \nu.$$

The nomenclature "$V - A$" comes from the fact that this interaction term, $\mathcal{L}_{\text{int.}}$, looks like the vector current minus the axial-vector current.

iii) Write down the invariant matrix element for the leading contribution to muon decay (i.e. the Feynman diagram you drew in part (d)i)).

Section 3

Consider now Fermi's effective four-fermion interaction Lagrangian

$$\mathcal{L}_F = -\frac{G_F}{\sqrt{2}} \left[\overline{\nu}_\mu \gamma^\mu (1-\gamma^5)\mu\right] \left[\overline{e}\gamma_\mu (1-\gamma^5)\nu_e\right].$$

(e) i) Draw the Feynman diagram that represents the leading order contribution to muon decay with this interaction Lagrangian.

ii) Write down the corresponding invariant matrix element for this diagram.

iii) Compare your expressions for the invariant matrix element in the full theory [your solution to question (d) iii)] and the effective theory [your solution to question (e) ii)] and show that these expressions are the same provided

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}.$$

Section 4

(f) i) Show that the invariant matrix element squared, in the effective theory, can be written as

$$|\mathcal{M}|^2 = 64 G_F^2 (p_\mu q_1^\mu)(k_\nu q_2^\nu),$$

where p^μ is the initial momentum of the muon, k^μ is the momentum of the final state electron, and $q_{1,2}$ are the momenta of the final state electron and muon neutrinos, respectively.

ii) Work in the muon rest frame and use the expression for the differential decay rate to three final state particles,

$$\mathrm{d}\Gamma(A \to p_1, p_2, p_3) = \frac{1}{2m_A} \left(\prod_{f=1}^{3} \frac{\mathrm{d}^3 \vec{p}_f}{(2\pi)^3 2E_{p_f}} \right) |\mathcal{M}(A \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)} \left(k_A - \sum_{f=1}^{3} p_f \right)$$

to show that, in the limit in which we can neglect the electron mass,

$$\mathrm{d}\Gamma = \frac{G_F^2}{12\pi^3 m_\mu}\Big(q^2(p\cdot k) + 2(q\cdot p)(q\cdot k)\Big)\, E\,\mathrm{d}E,$$

where $q^\mu = q_1^\mu + q_2^\mu$, and $E = E_k$ is the energy of the final state electron. You may find the integral

$$I^{\mu\nu}(q^2) = \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E_2}\delta^{(4)}(q - q_1 - q_2) q_1^\mu q_2^\nu = \frac{1}{6\pi(4\pi)^4}\Big(q^2 g^{\mu\nu} + 2q^\mu q^\nu\Big)$$

helpful.

iii) The kinematics of three-body final states means that the minimum final state electron energy is $E_{\mathrm{min}} = m_e \simeq 0$ and the maxium is $E_{\mathrm{max}} = m_\mu/2$. Integrate the differential decay rate over this kinematic region to show

$$\Gamma \simeq \frac{G_F^2 m_\mu^5}{192\pi^3}.$$

iv) Use the muon lifetime and mass [3]

$$\tau_\mu = 2.1969811(22)\times 10^{-6}\ \mathrm{s}, \qquad m_\mu = 105.6583745(24)\ \mathrm{MeV}$$

to obtain an estimate of G_F. Compare this to the current value from the Particle Data Group [3]

$$G_F = 1.1663787(6)\times 10^{-5}\ \mathrm{GeV}^{-2}.$$

Conclusion

(g) Summarise your work and give a brief overview of potential future steps or applications.

References

[1] Peskin, M. and Shroeder, D., An Introduction to Quantum Field Theory, CRC Press, 2019.

[2] Tong, D., Quantum Field Theory, Part III Maths Tripos Lecture Notes.

[3] Zyla, P.A. *et al.* (Particle Data Group), PTEP (2020) 083C01.