Quantum Field Theory I: PHYS 721
Problem Set 8: Solutions

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Overview

The questions in this problem set familiarise you with deriving Feynman rules from an example of an interacting theory, and writing down Feynman diagrams corresponding to time-ordered correlation functions. In many applications, this is a typical way to carry out calculations in QFT–given a Lagrangian, you derive the Feynman rules, write the Feynman diagrams that correspond to that process at a given order in perturbation theory (and then calculate the diagrams). There are two questions.

Question 1 [12]

In our mini-lectures we derived the Feynman rules for $\lambda\phi^4$ theory, which ended up being pretty simple. In fact, it is usually possible to just read off the Feynman rules directly from the Lagrangian. Consider the following Lagrangian[1],

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3,$$

which defines the so-called $\lambda\phi^3$ theory (very imaginatively named, as you can see).

(a) What is the mass dimension of the coupling λ in four spacetime dimensions? Is this coupling relevant, marginal, or irrelevant in four spacetime dimensions? In how many spacetime dimensions is λ dimensionless?

(b) Write out what you expect the relevant position-space Feynman rules to be. You do not need to provide a detailed derivation via Wick's theorem.

(c) Write out what you expect the relevant momentum-space Feynman rules to be. You do not need to provide a detailed derivation.

(d) Draw all the connected diagrams contributing, up to and including $\mathcal{O}(\lambda^2)$, to the:

(i) The two-point function

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle.$$

(ii) The three-point function

$$\langle\Omega|T\{\phi(x)\phi(y)\phi(z)\}|\Omega\rangle.$$

(iii) The four-point function

$$\langle\Omega|T\{\phi(w)\phi(x)\phi(y)\phi(z)\}|\Omega\rangle.$$

[1] Be aware! This is not exactly the same Lagrangian as we considered in class, or Schwartz Section 7.4!

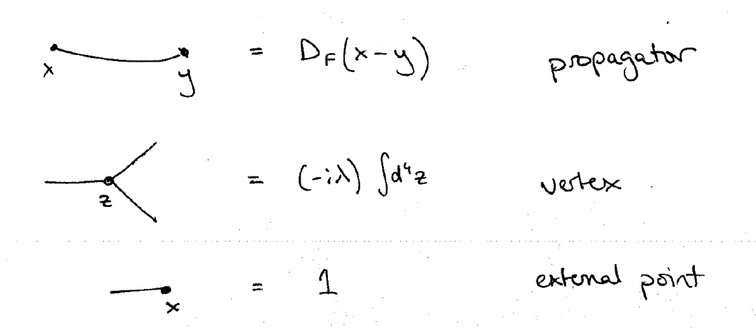


Figure 1: Position-space Feynman rules for the $\lambda\phi^3$ theory.

Solution 1

(a) We know that the Lagrangian density must have dimension four, because the four-integral of the Lagrangian density gives the action, which has units equal to $\hbar = 1$. Therefore we deduce that $[\phi] = 1$, so $[\phi^3] = 3$. This means that for $[\lambda\phi^3] = 4$, we must have $[\lambda] = 1$. Therefore this coupling is relevant in four spacetime dimensions.

In an arbitrary spacetime dimension d, we have $[\phi] = (d-2)/2$ (because $[m^2\phi^2] = d$), so $[\phi^3] = 3(d-2)/2$. We want to find d such that $[\lambda] = 0$. Therefore, we need to solve

$$0 + \frac{3(d-2)}{2} = d,$$

or $d = 6$. The coupling is dimensionless in six spacetime dimensions.

Note that the d-dimensional generalisation of the expression I used in class, λ/E^{4-n} is not λ/E^{d-n}. It is in fact $\lambda/E^{d-n/2(d-2)}$. This follows from the fact that the field dimension is $[\phi] = (d-2)/2$. One cannot assume $[\phi] = 1$, since this holds in only four dimensions.

(b) The position space Feynman rules are shown in Fig. 1.

(c) The momentum space Feynman rules are shown in Fig. 2.

(d) Diagrams below.

(i) The diagrams contributing to the two-point function are shown in Fig. 3.

(ii) The diagrams contributing to the three-point function are shown in Fig. 4.

(iii) The diagrams contributing to the four-point function are shown in Fig. 5.

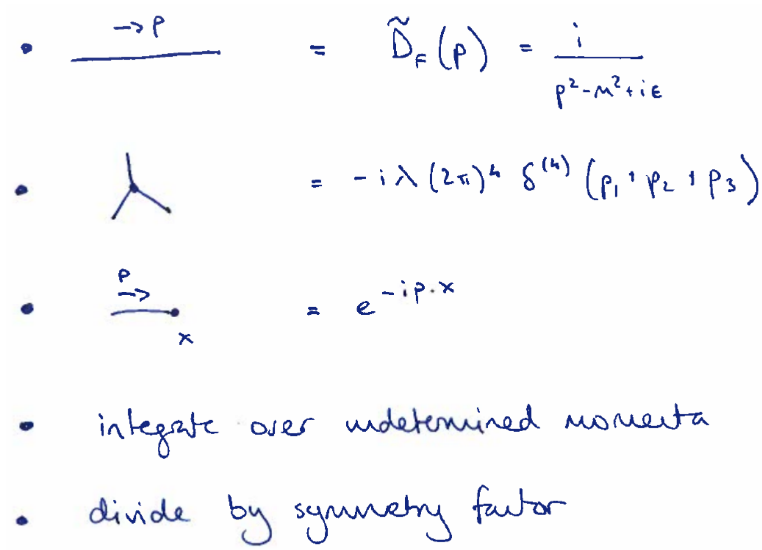


Figure 2: Momentum-space Feynman rules for the $\lambda\phi^3$ theory.

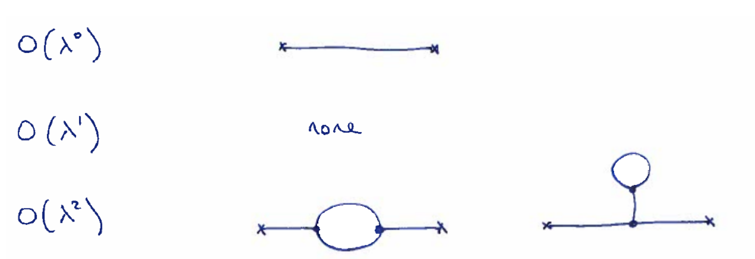


Figure 3: Connected diagrams contributing to the two-point function in the $\lambda\phi^3$ theory.

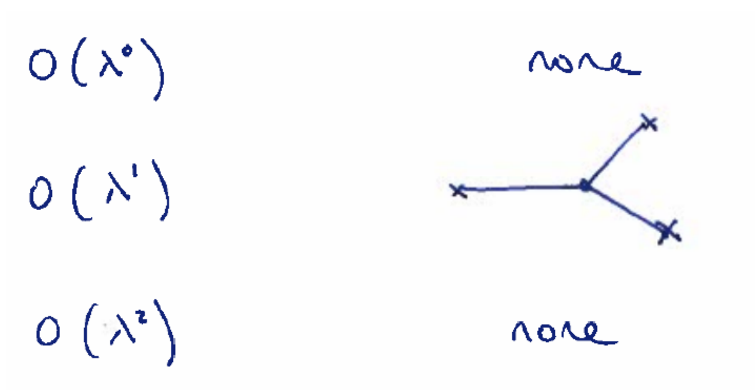


Figure 4: Connected diagrams contributing to the three-point function in the $\lambda\phi^3$ theory.

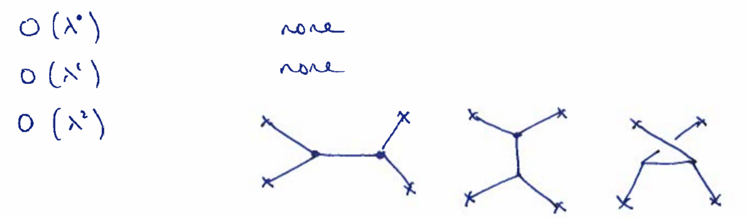


Figure 5: Connected diagrams contributing to the four-point function in the $\lambda\phi^3$ theory.

Question 2 **[8]**

Consider scalar Yukawa theory, defined by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - M^2 \psi^* \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - g\psi^*\psi\phi,$$

where ψ is a complex scalar field and ϕ is a real scalar field. Scalar Yukawa theory is a toy model for the low energy interactions of hadrons, such as pions and neutrons and protons. In reality, many hadrons are fermions, so the full Yukawa model (in which a scalar field couples to spinor fields) is a more realistic representation of hadron interactions at low energy. As the energy of the process increases, both of these models start to fail and the process must be calculated in the full theory of the strong interaction, quantum chromodynamics (which governs the behaviour of quarks and gluons within hadrons, and the interactions between hadrons).

(a) Draw the leading nontrivial (i.e. do not count the case of no scattering) Feynman diagrams for the following processes, and express the invariant matrix element for each process in terms of Mandelstam variables:

1. $\psi\psi$ scattering, $\psi\psi \to \psi\psi$;
2. $\psi\psi^*$ scattering, $\psi\psi^* \to \psi\psi^*$;
3. ϕ pair production, $\psi\psi^* \to \phi\phi$;
4. $\psi\phi$ scattering, $\psi\phi \to \psi\phi$.

(b) Draw the leading nontrivial (i.e. do not count the case of no scattering) Feynman diagrams for the following processes:

1. ϕ decay, $\phi \to \psi\psi^*$ (assuming $m > 2M$);
2. $\phi\phi$ scattering, $\phi\phi \to \phi\phi$.

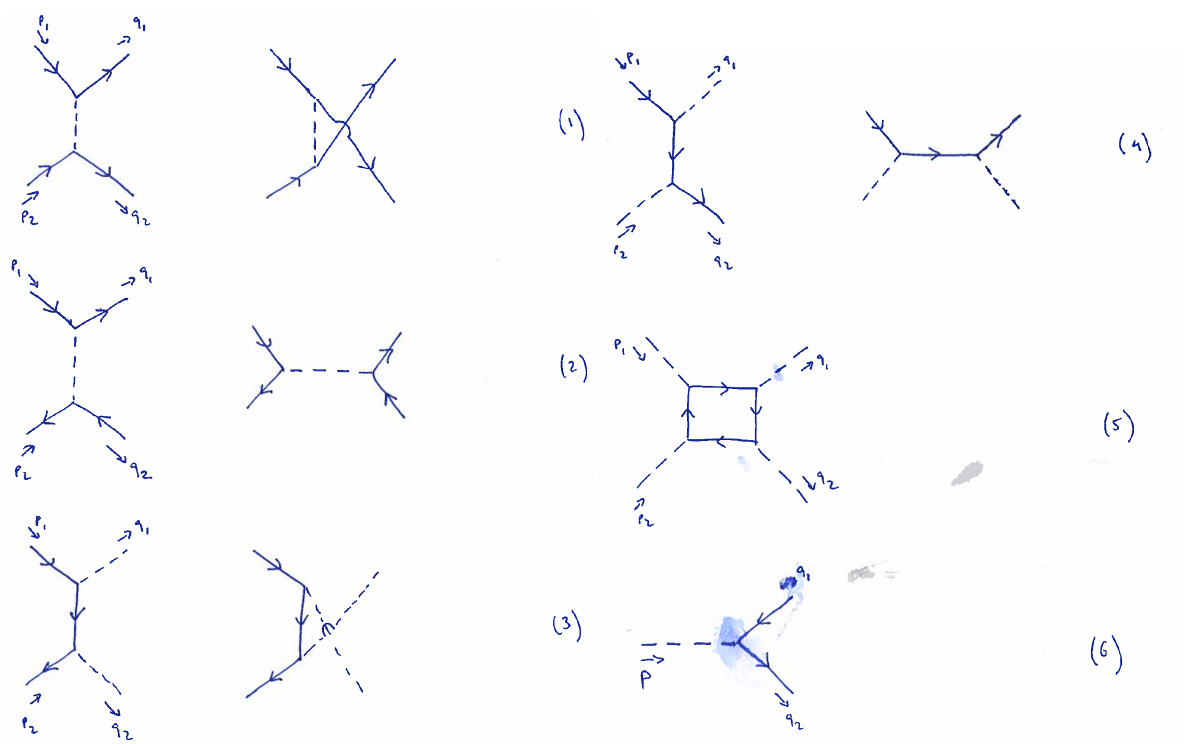
Solution 2

(a) The Feynman diagrams for these processes, and those for part (b), appear in Fig. 6. The invariant matrix elements for the processes in part (a) are

1. "Nucleon-nucleon" scattering,

$$i\mathcal{M} = -ig^2\left[\frac{1}{t-m^2} + \frac{1}{u-m^2}\right];$$

Figure 6: Feynman diagrams that contribute to each of the processes listed in the bullet points one to five, labelled by the appropriate number. Solid lines are ψ and $\overline{\psi}$ fields ("nucleons" and "antinucleons") and dashed lines are ϕ fields ("mesons"). The momenta in the right hand diagrams are understood to be the same as in the left hand diagrams.



2. "Nucleon-antinucleon" scattering,

$$i\mathcal{M} = -ig^2 \left[\frac{1}{t - m^2} + \frac{1}{s - m^2 + i\epsilon} \right];$$

3. "Meson" pair production,

$$i\mathcal{M} = -ig^2 \left[\frac{1}{t - M^2} + \frac{1}{u - M^2} \right];$$

4. "Nucleon-meson" scattering,

$$i\mathcal{M} = -ig^2 \left[\frac{1}{s - M^2 + i\epsilon} + \frac{1}{t - M^2} \right].$$