Quantum Field Theory I: PHYS 721
Problem Set 7: Solutions

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Overview

The questions in this problem introduce you to the *Mandelstam variables*, which describe two-body scattering, and then review the connection between spin and statistics. There are two questions.

Question 1 [10]

Consider a scattering process $A(p_1)A(p_2) \to A(q_1)A(q_2)$, involving only a single species of particle, A, with mass, m.

The entire process can be considered to occur in a plane, so we only need one angle and one energy variable to describe the process. It is simplest to use the centre-of-momentum frame, in which $\vec{p}_1 + \vec{p}_2 = \vec{q}_1 + \vec{q}_2 = 0$. Define the direction of $\vec{p}_1$ to be the z-axis and the direction of $\vec{q}_1$ to be at an angle $\theta_{\rm CM}$ with respect to the z-axis.

We can define three Lorentz-invariant kinematic variables, which are given by

$$s \equiv (p_1 + p_2)^2, \qquad t \equiv (q_1 - p_1)^2, \qquad \text{and} \qquad u \equiv (q_2 - p_1)^2.$$

These are the Lorentz invariant *Mandelstam variables*.

(a) Show that these variables are related by the condition

$$s + t + u = 4m^2,$$

so that there are only two independent variables.

(b) Show that these Mandelstam variables can be related to the centre-of-momentum frame variables by

$$\begin{aligned} s &= E_{\rm CM}^2, \\ t &= 2|\vec{p}_{\rm CM}|^2(\cos\theta_{\rm CM} - 1), \\ u &= -2|\vec{p}_{\rm CM}|^2(1 + \cos\theta_{\rm CM}), \end{aligned}$$

where

$$|\vec{p}_{\rm CM}| = \sqrt{\frac{E_{\rm CM}^2}{4} - m^2}$$

is the momentum of any of the particles in the centre-of-momentum frame.

(c) What are the ranges of the Mandelstam variables for physical scattering?

Solution 1

(a) In the center-of-momentum frame we have

$$\vec{p}_1 + \vec{p}_2 = \vec{q}_1 + \vec{q}_2 = 0,$$

and we can write the momenta as

$$\begin{aligned}
p_1^\mu &= (E_{\text{CM}}/2, 0, 0, |\vec{p}_{\text{CM}}|) \\
p_2^\mu &= (E_{\text{CM}}/2, 0, 0, -|\vec{p}_{\text{CM}}|) \\
q_1^\mu &= (E_{\text{CM}}/2, |\vec{p}_{\text{CM}}| \sin\theta_{\text{CM}}, 0, |\vec{p}_{\text{CM}}| \cos\theta_{\text{CM}}) \\
q_2^\mu &= (E_{\text{CM}}/2, -|\vec{p}_{\text{CM}}| \sin\theta_{\text{CM}}, 0, -|\vec{p}_{\text{CM}}| \cos\theta_{\text{CM}})
\end{aligned}$$

Then the Mandelstam variables are

$$\begin{aligned}
s &\equiv (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = 2m^2 + 2p_1 \cdot p_2, \\
t &\equiv (q_1 - p_1)^2 = q_1^2 + p_1^2 - 2p_1 \cdot q_1 = 2m^2 - 2p_1 \cdot q_1, \\
u &\equiv (q_2 - p_1)^2 = q_2^2 + p_1^2 - 2p_1 \cdot q_2 = 2m^2 - 2p_1 \cdot q_2.
\end{aligned}$$

The sum of these is then

$$\begin{aligned}
s + t + u &= 2m^2 + 2p_1 \cdot p_2 + 2m^2 - 2p_1 \cdot q_1 + 2m^2 - 2p_1 \cdot q_2 \\
&= 6m^2 + 2p_1 \cdot p_2 - 2p_1 \cdot (q_1 + q_2) \\
&= 6m^2 + 2p_1 \cdot p_2 - 2p_1 \cdot (p_1 + p_2) \\
&= 6m^2 - 2p_1^2 \\
&= 4m^2.
\end{aligned}$$

(b) In the centre-of-momentum frame we have

$$\begin{aligned}
(p_1 + p_2)^\mu &= (E_{\text{CM}}, 0, 0, 0), \\
(q_1 - p_1)^\mu &= (0, |\vec{p}_{\text{CM}}| \sin\theta_{\text{CM}}, 0, \vec{p}_{\text{CM}}(\cos\theta_{\text{CM}} - 1)),
\end{aligned}$$

so that

$$\begin{aligned}
s &= (p_1 + p_2)^2 = E_{\text{CM}}^2, \\
t &= (q_1 - p_1)^2 = -(|\vec{p}_{\text{CM}}|^2 \sin^2\theta_{\text{CM}} + |\vec{p}_{\text{CM}}|^2 (\cos\theta_{\text{CM}} - 1)^2) = 2|\vec{p}_{\text{CM}}|^2 (\cos\theta_{\text{CM}} - 1).
\end{aligned}$$

Then we can most easily calculate u via

$$\begin{aligned}
u &= 4m^2 - s - t \\
&= 4m^2 - 4(|\vec{p}_{\text{CM}}|^2 + m^2) + 2|\vec{p}_{\text{CM}}|^2 (1 - \cos\theta_{\text{CM}}) \\
&= -2|\vec{p}_{\text{CM}}|^2 (1 + \cos\theta_{\text{CM}}).
\end{aligned}$$

(c) The minimum energy in the centre-of-momentum frame is $E_{\text{CM}} = 2m$, so for physical scattering we must have $s \geq 4m^2$. The cosine function is restricted to the range $[-1, 1]$, so $-4|\vec{p}_{\text{CM}}|^2 \leq t, u \leq 0$, although u and t do not have the same endpoints as a function of θ_{CM} (for example $\theta_{\text{CM}} = 0$ means $t = 0$, which is forward scattering, but $u = -4|\vec{p}_{\text{CM}}|^2$).

Question 2 **[10]**

Write 250 to 300 words[1] discussing the relationship between spin and statistics in quantum field theory. You should illustrate this with at least two examples. Your response should be written in full sentences and addressed as a popular science article (think Scientific American, New Scientist or Discover), or blog (think the science section of Ars Technica, Preposterous Universe or Science Alert). You will be graded using the following rubric:

Aspect	**Points**	**If you:**
Physics	4	Correctly characterise: relationship between particle spin and particle statistics; consequences for commutation relations of fields; consequences for two-point functions.
	2	Correctly characterise most of these points, but not all; or describe all points, but miss key information.
	0	Completely misconstrue the relationships.
Examples	4	Provide at least two examples.
	2	Provide one example; or two examples, but not correctly.
	0	Give no examples
Audience	2	Correctly gauge the understanding of the audience, including defining and illustrating terms as appropriate.
	1	Give a too-technical or too-simplistic explanation.

Solution 2

The rubric is the solution. Keep in mind the science writing adage, from Stephen Hawking:

> *Someone told me that each equation I included in the book [A Brief History of Time] would halve its sales.*

[1] I will count, but perhaps not too carefully.