Quantum Field Theory I: PHYS 721
Problem Set 4: Solutions

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Overview

The questions in this problem set give you some practice at manipulating Lorentz transformation matrices and relating those explicit expressions to their effects on fields. There are two questions.

Question 1 **8pts**

(a) Anti-clockwise rotations in a two-dimensional plane can be expressed as

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Use this form to determine the corresponding generator J, defined through

$$R = e^{-i\theta J}.$$

Show that this generator satisfies

$$J^n = \begin{cases} 1 & \text{for } n \text{ even} \\ J & \text{for } n \text{ odd} \end{cases}$$

(b) Generalise this result to three dimensions by writing down matrices R_x, R_y, and R_z, representing anti-clockwise rotations around the x, y, and z axes respectively. Determine the corresponding generators, J_i, from

$$R_i = e^{-i\theta J_i},$$

where i runs over the three spatial indices $\{x, y, z\}$, and deduce the Lie algebra of the rotation group (i.e. find the commutator $[J_i, J_j]$).

Solution 1

(a) For an infinitesimal rotation, $\delta\theta$, we can express the rotation matrix as

$$\begin{aligned} R &= e^{-i\delta\theta J} \\ &\simeq 1 - i\delta\theta J + \mathcal{O}((\delta\theta)^2). \end{aligned}$$

Expanding our expression for R gives

$$R \simeq \begin{pmatrix} 1 + \mathcal{O}((\delta\theta)^2) & -i\delta\theta \\ i\delta\theta & 1 + \mathcal{O}((\delta\theta)^2) \end{pmatrix}$$

from which we deduce that

$$J = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

It is then straightforward to see that $J^2 = 1$, and therefore that

$$J^n = \begin{cases} 1 & \text{for } n \text{ even} \\ J & \text{for } n \text{ odd} \end{cases}$$

(b) The rotation matrices in three dimensions are

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$
$$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix},$$
$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From the infinitesimal expansion of these, we deduce

$$J_x = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$J_y = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$
$$J_z = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To determine the Lie algebra, we can consider some examples

$$\begin{aligned} [J_x, J_y] &= J_x J_y - J_y J_x \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= iJ_z. \end{aligned}$$

Similarly we can show that

$$
\begin{aligned}
[J_x, J_z] &= J_x J_z - J_z J_x \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= -iJ_y
\end{aligned}
$$

Since we can easily show that $[J_x, J_x] = 0$, we can deduce that the Lie algebra is

$$[J_i, J_j] = i\epsilon_{ijk} J_k.$$

Question 2 **12pts**

(a) Consider an anti-clockwise rotation of $\theta = \pi/20$ about the z-axis. Write down explicit expressions for the:

1. four-dimensional matrix, Λ^μ_ν, that corresponds to this Lorentz transformation;
2. generator of the Lorentz group that corresponds to this rotation in the following representations:
   i. scalar;
   ii. vector;
   iii. spinor.

(b) Using these results, derive the effect of this transformation on the following fields:

1. scalar, $\phi(x)$;
2. vector, $A^\mu(x) = (A^0, A^1, A^2, A^3)$;
3. spinor, $u^s(p) = m(\xi^s, \xi^s)$, where $\xi^s = (1, 0)$ is a left-handed two-component Weyl spinor (which could represent, for example, an electron with spin up in the z-direction).

Express your results in terms of the original component or components of each field.

Solution 2

(a) We have:

1.
$$\Lambda^\mu_{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\pi/20 & -\sin\pi/20 & 0 \\ 0 & \sin\pi/20 & \cos\pi/20 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.9876 & -0.1564 & 0 \\ 0 & 0.1564 & 0.9876 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. The generators are

   i. The scalar representation is the identity representation, so the generator is

$$\overset{0}{J}{}^{\mu\nu} = 0$$

   ii. In general the vector representation is given by

$$(\overset{1}{J}{}^{\mu\nu})_{\alpha\beta} = (V^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)$$

   and so in this case we have

$$(\overset{1}{J}{}^{12})_{\alpha\beta} = i\left(\delta^1_\alpha \delta^2_\beta - \delta^1_\beta \delta^2_\alpha\right).$$

   Note

$$(\overset{1}{J}{}^{12})^\alpha_{\ \beta} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   iii. In general the spinor representation is given by

$$\overset{\frac{1}{2}}{J}{}^{\mu\nu} = S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

   and so in this case we have (in the Weyl representation of the gamma matrices)

$$\begin{aligned}
(\overset{\frac{1}{2}}{J}{}^{12})^\alpha_{\ \beta} &= \frac{i}{4}(\gamma^1\gamma^2 - \gamma^2\gamma^1)^\alpha_{\ \beta} \\
&= \frac{i}{4}\left[\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\right. \\
&\qquad \left. - \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}$$

(b) This action of a Lorentz transformation on fields is the exponential of the generator, multiplied by the appropriate parameters (in this case, the angle $\pi/20$). Thus we have

1. For the scalar

$$\phi'(\Lambda x) = \phi(x).$$

2. A vector field transforms as

$$A'^{\mu}(\Lambda x) = (\Lambda_V)^{\mu}_{\nu} A^{\nu}(x),$$

where

$$(\Lambda_V)_{\alpha\beta} = \exp\left[-i\theta_{\mu\nu}(\overset{1}{J}{}^{\mu\nu})_{\alpha\beta}\right].$$

Note that this choice of sign convention is chosen to match the first part of the problem set, rather than Schwartz. In this case we have

$$(\Lambda_V)^{\alpha}_{\beta} = \exp\left[-i\theta_{12}(\overset{1}{J}{}^{12})^{\alpha}_{\beta} - i\theta_{21}(\overset{1}{J}{}^{21})^{\alpha}_{\beta}\right],$$

where $2\theta_{12} = \pi/20$. Note, too, that this part of the question is not the same convention as the first question in the problem set. Thus, expanding the exponential, we have

$$\begin{aligned}
A'^{\mu}(\Lambda x) &= \left[\delta^{\mu}_{\nu} - \frac{i\pi}{20}(\overset{1}{J}{}^{12})^{\mu}_{\nu} + \cdots\right] A^{\nu}(x) \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{i\pi}{20}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\pi/20 & 0 \\ 0 & \pi/20 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \\
&= \begin{pmatrix} A^0 \\ A^1 - \pi/20 A^2 \\ \pi/20 A^1 + A^2 \\ A^3 \end{pmatrix}.
\end{aligned}$$

3. We now repeat this procedure for the spinor representation. A spinor field transforms as

$$u'^{s}(p) = \Lambda_S u^s(\Lambda p),$$

where matrix multiplication is implicit and

$$(\Lambda_S)^{\alpha}_{\beta} = \exp\left[-i\theta_{\mu\nu}(\overset{\frac{1}{2}}{J}{}^{\mu\nu})^{\alpha}_{\beta}\right].$$

In our example

$$\Lambda_S = \exp\left[-i\theta_{12}\overset{\frac{1}{2}}{J}{}^{12} - i\theta_{21}\overset{\frac{1}{2}}{J}{}^{21}\right],$$

so

$$
\begin{aligned}
(u'^s)^\mu(p) &= \left[\delta^\mu_\nu - \frac{i\pi}{20}(\overset{\frac{1}{2}}{J}{}^{12})^\mu_\nu + \cdots\right](u^s(p))^\nu \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{i\pi}{20}\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right] m \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 - i\pi/40 & 0 & 0 & 0 \\ 0 & 1 + i\pi/40 & 0 & 0 \\ 0 & 0 & 1 - i\pi/40 & 0 \\ 0 & 0 & 0 & 1 + \pi/40 \end{pmatrix} m \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= m \begin{pmatrix} 1 - i\pi/40 \\ 0 \\ 1 - i\pi/40 \\ 0 \end{pmatrix}
\end{aligned}
$$

Note this extra factor of one half in the spinor representation, compared to the vector representation. This corresponds to the fact that spinors require a rotation of 4π to return to themselves! Note also that under rotations, both the left-handed and right-handed components of the spinor transform in the same way. This would not be true if we had chosen to study a boost.