Quantum Field Theory I: PHYS 721
Problem Set 2: Solutions

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Overview

The questions in this problem give you some practice at manipulating groups and representations of the Lorentz group.

Question 1 **[8 pts]**

(a) Show that the translation operator

$$\widehat{U}(\mathbf{a})|\psi(\mathbf{x})\rangle = |\psi(\mathbf{x}+\mathbf{a})\rangle$$

is unitary.

(b) Show that translations act on the position operator $\widehat{\mathbf{x}}$ as

$$\widehat{U}^\dagger(\mathbf{a})\widehat{\mathbf{x}}\,\widehat{U}(\mathbf{a}) = \widehat{\mathbf{x}} + \mathbf{a}.$$

(c) Show that translations satisfy the properties of a group.

Solution 1

(a) We require that the translation operator does not change the probability density of the state, so

$$\langle\psi(\mathbf{x}+\mathbf{a})|\psi(\mathbf{x}+\mathbf{a})\rangle = \langle\psi(\mathbf{x})|U^\dagger(\mathbf{a})U(\mathbf{a})|\psi(\mathbf{x})\rangle = \langle\psi(\mathbf{x})|\psi(\mathbf{x})\rangle,$$

which entails $U^\dagger(\mathbf{a})U(\mathbf{a}) = 1$, as required.

(b) From the definition of the translation operator we have

$$\begin{aligned}\widehat{\mathbf{x}}\widehat{U}(\mathbf{a})|\mathbf{x}\rangle &= \widehat{\mathbf{x}}|\mathbf{x}+\mathbf{a}\rangle \\ &= (\mathbf{x}+\mathbf{a})|\mathbf{x}+\mathbf{a}\rangle,\end{aligned}$$

and so

$$\begin{aligned}\widehat{U}^\dagger(\mathbf{a})\widehat{\mathbf{x}}\widehat{U}(\mathbf{a})|\mathbf{x}\rangle &= (\mathbf{x}+\mathbf{a})\widehat{U}^\dagger(\mathbf{a})|\mathbf{x}+\mathbf{a}\rangle \\ &= (\mathbf{x}+\mathbf{a})|\mathbf{x}\rangle.\end{aligned}$$

Therefore we deduce

$$\widehat{U}^\dagger(\mathbf{a})\widehat{\mathbf{x}}\,\widehat{U}(\mathbf{a}) = \widehat{\mathbf{x}} + \mathbf{a},$$

as required.

(c) The group conditions are clearly satisfied:

1. Two translations are always equal to another translation $\widehat{U}(\mathbf{a}) \cdot \widehat{U}(\mathbf{b}) = \widehat{U}(\mathbf{a}+\mathbf{b})$, so we have closure under a group operation (composition of translations).
2. There is an identity element (translation by the zero-vector), $1 = \widehat{U}(\mathbf{0})$.
3. There is clearly an inverse of $\widehat{U}(\mathbf{a})$, which is $\widehat{U}^\dagger(\mathbf{a}) = \widehat{U}(-\mathbf{a})$.
4. We know two translations equal another translation, and it doesn't matter which order of operation we apply them

$$\widehat{U}(\mathbf{a}) \cdot \left(\widehat{U}(\mathbf{b}) \cdot \widehat{U}(\mathbf{c})\right) = \left(\widehat{U}(\mathbf{a}) \cdot \widehat{U}(\mathbf{b})\right) \cdot \widehat{U}(\mathbf{c}),$$

   since in both cases the outcome is equivalent to $\widehat{U}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. So we have associativity.

Question 2 **[12 pts]**

(a) Show that the Lorentz group generators in the vector representation, given by

$$(\overset{1}{J}{}^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)$$

satisfy the Lorentz Lie algebra, that is, show that

$$[\overset{1}{J}{}^{\mu\nu}, \overset{1}{J}{}^{\rho\sigma}] = i\left(g^{\nu\rho}\,\overset{1}{J}{}^{\mu\sigma} - g^{\mu\rho}\,\overset{1}{J}{}^{\nu\sigma} - g^{\nu\sigma}\,\overset{1}{J}{}^{\mu\rho} + g^{\mu\sigma}\,\overset{1}{J}{}^{\nu\rho}\right).$$

(b) Show that the Lorentz group generators in the spinor representation, given by

$$\overset{\frac{1}{2}}{J}{}^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu],$$

satisfy

$$[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^\rho] = i\left(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}\right).$$

(c) Show the Lorentz group generators in the spinor representation satisfy the Lorentz Lie algebra, that is, show that

$$[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \overset{\frac{1}{2}}{J}{}^{\rho\sigma}] = i\left(g^{\nu\rho}\,\overset{\frac{1}{2}}{J}{}^{\mu\sigma} - g^{\mu\rho}\,\overset{\frac{1}{2}}{J}{}^{\nu\sigma} - g^{\nu\sigma}\,\overset{\frac{1}{2}}{J}{}^{\mu\rho} + g^{\mu\sigma}\,\overset{\frac{1}{2}}{J}{}^{\nu\rho}\right).$$

You may find the result from part (b) helpful.

Solution 2

(a) For this we need

$$\begin{aligned}(J^{\mu\nu})_{\alpha\gamma}(J^{\rho\sigma})^{\gamma\beta} &= -\left(\delta^\mu_\alpha\delta^\nu_\gamma - \delta^\nu_\alpha\delta^\mu_\gamma\right)\left(\delta^{\rho\gamma}\delta^{\sigma\beta} - \delta^{\rho\beta}\delta^{\sigma\gamma}\right)\\ &= -\delta^\mu_\alpha\delta^{\rho\nu}\delta^{\sigma\beta} + \delta^\mu_\alpha\delta^{\rho\beta}\delta^{\sigma\nu} + \delta^\nu_\alpha\delta^{\rho\mu}\delta^{\sigma\beta} - \delta^\nu_\alpha\delta^{\rho\beta}\delta^{\sigma\mu}\end{aligned}$$

and

$$\begin{aligned}(J^{\rho\sigma})_{\alpha\gamma}(J^{\mu\nu})^{\gamma\beta} &= -\left(\delta^\rho_\alpha\delta^\sigma_\gamma - \delta^\rho_\gamma\delta^\sigma_\alpha\right)\left(\delta^{\mu\gamma}\delta^{\nu\beta} - \delta^{\mu\beta}\delta^{\nu\gamma}\right)\\ &= -\delta^\rho_\alpha\delta^{\sigma\mu}\delta^{\nu\beta} + \delta^\rho_\alpha\delta^{\mu\beta}\delta^{\sigma\nu} + \delta^\sigma_\alpha\delta^{\rho\mu}\delta^{\nu\beta} - \delta^\sigma_\alpha\delta^{\rho\nu}\delta^{\mu\beta}.\end{aligned}$$

Putting these together, we have

$$\begin{aligned}[J^{\mu\nu}, J^{\rho\sigma}]^\beta_\alpha &= \delta^{\nu\rho}\left(-\delta^\mu_\alpha\delta^{\sigma\beta} + \delta^\sigma_\alpha\delta^{\mu\beta}\right) + \delta^{\nu\sigma}\left(\delta^\mu_\alpha\delta^{\rho\beta} - \delta^\rho_\alpha\delta^{\mu\beta}\right)\\ &\quad + \delta^{\mu\rho}\left(\delta^\nu_\alpha\delta^{\sigma\beta} - \delta^\sigma_\alpha\delta^{\nu\beta}\right) - \delta^{\mu\sigma}\left(\delta^\nu_\alpha\delta^{\rho\beta} - \delta^\rho_\alpha\delta^{\nu\beta}\right)\\ &= ig^{\nu\rho}i\left(\delta^\mu_\alpha\delta^{\sigma\beta} - \delta^\sigma_\alpha\delta^{\mu\beta}\right) - ig^{\nu\sigma}i\left(\delta^\mu_\alpha\delta^{\rho\beta} - \delta^\rho_\alpha\delta^{\mu\beta}\right)\\ &\quad - ig^{\mu\rho}i\left(\delta^\nu_\alpha\delta^{\sigma\beta} - \delta^\sigma_\alpha\delta^{\nu\beta}\right) + ig^{\mu\sigma}i\left(\delta^\nu_\alpha\delta^{\rho\beta} - \delta^\rho_\alpha\delta^{\nu\beta}\right)\\ &= ig^{\nu\rho}(J^{\mu\nu})^\beta_\alpha - ig^{\nu\sigma}(J^{\mu\rho})^\beta_\alpha - ig^{\mu\rho}(J^{\nu\sigma})^\beta_\alpha + ig^{\mu\sigma}(J^{\nu\rho})^\beta_\alpha.\end{aligned}$$

Note that we have some useful properties of the delta function with raised and lowered indices. We start from the definition

$$\delta^\mu{}_\mu = +1,$$

for all μ (with no summation over μ in this case). Then it follows that

$$\delta^{\mu\nu} = \delta^\mu{}_\rho g^{\rho\nu} = g^{\mu\nu}$$

In particular we have

$$\begin{aligned}\delta^{00} &= \delta^0{}_0 g^{00} = +1,\\ \delta^{ij} &= \delta^i{}_k g^{kj} = -1\end{aligned}$$

for $i = j$ $(= k)$. All other values are zero.

Similarly we have

$$\delta_{\mu\nu} = \delta^\rho{}_\nu g_{\rho\mu} = g^{\rho\sigma} g_{\rho\mu} g_{\sigma\nu},$$

where we've used

$$g^{\rho\sigma} g_{\sigma\nu} = \delta^\rho{}_\nu.$$

Thus

$$\delta_{00} = g^{00} g_{00} g_{00} = +1,$$

and, with no summation over i implied,

$$\delta_{ii} = g^{jk} g_{ij} g_{ik} = (-1)^3 = -1.$$

So we also have

$$\delta_{\mu\nu} = g_{\mu\nu}.$$

(b) Using the definition of the spinor representation of the Lorentz group, we have

$$\begin{aligned}[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^\rho] &= \frac{i}{4}[[\gamma^\mu, \gamma^\nu], \gamma^\rho] \\ &= \frac{i}{4}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^\rho].\end{aligned}$$

We can assume that $\mu \neq \nu$, since otherwise the lefthand side is trivially zero. Then we have

$$\begin{aligned}[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^\rho] &= \frac{i}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] \\ &= \frac{i}{2}\left(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu\right) \\ &= \frac{i}{2}\left(\gamma^\mu\left(\{\gamma^\nu, \gamma^\rho\} - \gamma^\rho\gamma^\nu\right) - \left(\{\gamma^\rho, \gamma^\mu\} - \gamma^\mu\gamma^\rho\right)\gamma^\nu\right) \\ &= \frac{i}{2}\left(\gamma^\mu 2g^{\nu\rho} - 2g^{\rho\nu}\gamma^\mu - \gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\mu\gamma^\rho\gamma^\nu\right) \\ &= i\left(\gamma^\mu g^{\nu\rho} - \gamma^\mu g^{\rho\nu}\right).\end{aligned}$$

(c) To show that spinor representation satisfies the Lie algebra of the Lorentz group, we use the result of (b). Plugging in the definition of the spinor representation, and taking $\rho \neq \sigma$, we have

$$\begin{aligned}[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \overset{\frac{1}{2}}{J}{}^{\rho\sigma}] &= \frac{i}{2}[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^\rho\gamma^\sigma] \\ &= \frac{i}{2}\left([\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^\rho]\gamma^\sigma + \gamma^\rho[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^\sigma]\right).\end{aligned}$$

Now we use part (c) to write this as

$$\begin{aligned}[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \overset{\frac{1}{2}}{J}{}^{\rho\sigma}] &= \frac{i}{2}\left(i\left(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\rho\mu}\right)\gamma^\sigma + \gamma^\rho\, i\left(\gamma^\mu g^{\nu\sigma} - \gamma^\nu g^{\sigma\mu}\right)\right) \\ &= \frac{i}{2}\left(i\gamma^\mu\gamma^\sigma g^{\nu\rho} - i\gamma^\nu\gamma^\sigma g^{\rho\mu} + i\gamma^\rho\gamma^\mu g^{\nu\sigma} - \gamma^\rho\gamma^\nu g^{\sigma\mu}\right).\end{aligned} \tag{1}$$

Now we recall that

$$\overset{\frac{1}{2}}{J}^{\mu\nu} = \frac{i}{2}\gamma^\mu\gamma^\nu - \frac{i}{2}g^{\mu\nu},$$

or, rearranging this, that

$$i\gamma^\mu\gamma^\nu = 2\overset{\frac{1}{2}}{J}^{\mu\nu} + ig^{\mu\nu}.$$

Using this in Equation (1), we have

$$\begin{aligned}
[\overset{\frac{1}{2}}{J}^{\mu\nu}, \overset{\frac{1}{2}}{J}^{\rho\sigma}] &= \frac{i}{2}\Big(\Big(2\overset{\frac{1}{2}}{J}^{\mu\sigma} + ig^{\mu\sigma}\Big)g^{\nu\rho} - i\Big(2\overset{\frac{1}{2}}{J}^{\nu\sigma} + ig^{\nu\sigma}\Big)g^{\rho\mu} \\
&\qquad + i\Big(2\overset{\frac{1}{2}}{J}^{\rho\mu} + ig^{\rho\mu}\Big)g^{\nu\sigma} - \Big(2\overset{\frac{1}{2}}{J}^{\rho\nu} + ig^{\rho\nu}\Big)g^{\sigma\mu}\Big) \\
&= i\Big(\overset{\frac{1}{2}}{J}^{\mu\sigma} g^{\nu\rho} - \overset{\frac{1}{2}}{J}^{\nu\sigma} g^{\rho\mu} - \overset{\frac{1}{2}}{J}^{\rho\nu} g^{\sigma\mu} + \overset{\frac{1}{2}}{J}^{\rho\mu} g^{\nu\sigma} \\
&\qquad + \frac{i}{2}\left(g^{\mu\sigma}g^{\nu\rho} - g^{\nu\sigma}g^{\rho\mu} + g^{\rho\mu}g^{\nu\sigma} - g^{\rho\nu}g^{\sigma\mu}\right) \\
&= i\Big(\overset{\frac{1}{2}}{J}^{\mu\sigma} g^{\nu\rho} - \overset{\frac{1}{2}}{J}^{\nu\sigma} g^{\rho\mu} - \overset{\frac{1}{2}}{J}^{\rho\nu} g^{\sigma\mu} + \overset{\frac{1}{2}}{J}^{\rho\mu} g^{\nu\sigma}\Big).
\end{aligned}$$