Quantum Field Theory I: PHYS 721
Problem Set 5

Chris Monahan

The questions in this problem set give you some practice at manipulating quantised operators and reinforce the close relationship between symmetries and conservation laws. There are two questions.

Question 1 **8pts**

(a) Use the expressions for the Hamiltonian and the momentum operator

$$H_0 = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} E_k a^\dagger(\vec{k}) a(\vec{k}),$$
$$P^i = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} k^i a^\dagger(\vec{k}) a(\vec{k})$$

for a free, real scalar field, $\phi(x)$, to show that the four-vector (H, P^i) generates spacetime translations

$$\phi(x) = e^{i(Ht - \vec{P}\cdot\vec{x})}\phi(0)e^{-i(Ht-\vec{P}\cdot\vec{x})}.$$

This is an example of a very general phenomenon: the conserved charge due to a symmetry generates the corresponding symmetry transformation on the fields.

[Hint: You will need to consider expressions of the form $e^{iH_0t}a(\vec{k})e^{-iH_0t}$.]

(b) Use this result to show that

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi(x-y)\phi(0)|0\rangle.$$

Question 2 [based on Peskin and Shroeder 2.2] **[12]**

In this question we study the quantum field theory of a complex scalar field, defined by the action

$$\widehat{S} = \int \mathrm{d}^4x \left(\partial_\mu \widehat{\phi}^* \partial^\mu \widehat{\phi} - m^2 \widehat{\phi}^* \widehat{\phi}\right). \tag{1}$$

We could choose to analyse this theory by treating the real and imaginary parts of the complex field $\widehat{\phi}$ as independent dynamical variables, but it is easier to instead choose $\widehat{\phi}$ and $\widehat{\phi}^*$ as the basic independent variables.

(a) Find the conjugate momenta of $\widehat{\phi}(x)$ and $\widehat{\phi}^*(x)$ and the corresponding canonical commutation relations.

(b) Show that the Hamiltonian is

$$\widehat{H} = \int \mathrm{d}^3\vec{x} \left(\widehat{\pi}^* \widehat{\pi} + \vec{\nabla}\widehat{\phi}^* \cdot \vec{\nabla}\widehat{\phi} + m^2 \widehat{\phi}^* \widehat{\phi} \right). \tag{2}$$

Determine the Heisenberg equation of motion for $\widehat{\phi}(x)$ and show that it leads to the Klein-Gordon equation.

(c) Write the Hamiltonian in terms of creation and annihilation operators[1] and show that the theory contains two sets of particles of mass m.

(d) Rewrite the conserved charge

$$\widehat{Q} = \frac{i}{2} \int \mathrm{d}^3\vec{x} \left(\widehat{\phi}^* \widehat{\pi}^* - \widehat{\pi} \widehat{\phi} \right) \tag{3}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

[1]Think about why the creation and annihilation operators were complex conjugates of each other in the real scalar field case. Does that have to be true in this case?