Quantum Field Theory I: PHYS 721
Problem Set 4

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The questions in this problem set give you some practice at manipulating Lorentz transformation matrices and relating those explicit expressions to their effects on fields. There are two questions.

Question 1 **8pts**

(a) Anti-clockwise rotations in a two-dimensional plane can be expressed as

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Use this form to determine the corresponding generator J, defined through

$$R = e^{-i\theta J}.$$

Show that this generator satisfies

$$J^n = \begin{cases} 1 & \text{for } n \text{ even} \\ J & \text{for } n \text{ odd} \end{cases}$$

(b) Generalise this result to three dimensions by writing down matrices R_x, R_y, and R_z, representing anti-clockwise rotations around the x, y, and z axes respectively. Determine the corresponding generators, J_i, from

$$R_i = e^{-i\theta J_i},$$

where i runs over the three spatial indices $\{x, y, z\}$, and deduce the Lie algebra of the rotation group (i.e. find the commutator $[J_i, J_j]$).

Question 2 **12pts**

(a) Consider an anti-clockwise rotation of $\theta = \pi/20$ about the z-axis. Write down explicit expressions for the:

1. four-dimensional matrix, Λ^{μ}_{ν}, that corresponds to this Lorentz transformation;
2. generator of the Lorentz group that corresponds to this rotation in the following representations:
   i. scalar;
   ii. vector;
   iii. spinor.

(b) Using these results, derive the effect of this transformation on the following fields:

1. scalar, $\phi(x)$;
2. vector, $A^{\mu}(x) = (A^0, A^1, A^2, A^3)$;
3. spinor, $u^s(p) = m(\xi^s, \xi^s)$, where $\xi^s = (1, 0)$ is a left-handed two-component Weyl spinor (which could represent, for example, an electron with spin up in the z-direction).

Express your results in terms of the original component or components of each field.