Quantum Field Theory I: PHYS 721
Problem Set 2

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The questions in this problem give you some practice at manipulating groups and representations of the Lorentz group.

Question 1 **[8 pts]**

(a) Show that the translation operator

$$\widehat{U}(\mathbf{a})|\psi(\mathbf{x})\rangle = |\psi(\mathbf{x}+\mathbf{a})\rangle$$

is unitary.

(b) Show that translations act on the position operator $\widehat{\mathbf{x}}$ as

$$\widehat{U}^\dagger(\mathbf{a})\widehat{\mathbf{x}}\,\widehat{U}(\mathbf{a}) = \widehat{\mathbf{x}} + \mathbf{a}.$$

(c) Show that translations satisfy the properties of a group.

Question 2 **[12 pts]**

(a) Show that the Lorentz group generators in the vector representation, given by

$$(\overset{1}{J}{}^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right)$$

satisfy the Lorentz Lie algebra, that is, show that

$$[\overset{1}{J}{}^{\mu\nu}, \overset{1}{J}{}^{\rho\sigma}] = i\left(g^{\nu\rho}\,\overset{1}{J}{}^{\mu\sigma} - g^{\mu\rho}\,\overset{1}{J}{}^{\nu\sigma} - g^{\nu\sigma}\,\overset{1}{J}{}^{\mu\rho} + g^{\mu\sigma}\,\overset{1}{J}{}^{\nu\rho}\right).$$

(b) Show that the Lorentz group generators in the spinor representation, given by

$$\overset{\frac{1}{2}}{J}{}^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu],$$

satisfy

$$[\overset{\frac{1}{2}}{J}{}^{\mu\nu}, \gamma^\rho] = i\left(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}\right).$$

(c) Show the Lorentz group generators in the spinor representation satisfy the Lorentz Lie algebra, that is, show that

$$[\overset{\frac{1}{2}}{J}^{\mu\nu}, \overset{\frac{1}{2}}{J}^{\rho\sigma}] = i\left(g^{\nu\rho}\,\overset{\frac{1}{2}}{J}^{\mu\sigma} - g^{\mu\rho}\,\overset{\frac{1}{2}}{J}^{\nu\sigma} - g^{\nu\sigma}\,\overset{\frac{1}{2}}{J}^{\mu\rho} + g^{\mu\sigma}\,\overset{\frac{1}{2}}{J}^{\nu\rho}\right).$$

You may find the result from part (b) helpful.