Learning outcomes - quantising free fields

You will be able to:

- relate simple harmonic oscillator excitations to particles
- write down equal-time commutation relations for scalar fields and creation and annihilation operators
- express operators in terms of creation and annihilation operators
- manipulate fields and operators through application of commutation relations
- define normal ordering and motivate the procedure
- explain why spinor fields satisfy anticommutation relations
- write down commutation relations for a vector field in Coulomb gauge
- define quantum fields as operator-valued functions of spacetime that transform under irreducible representations of the Lorentz group.

- relate causality in quantum field theory to time-ordering
- calculate Green functions of the Klein-Gordon operator

* Quick Quiz

Quantising free fields

We have spent a lot of time analysing the Lorentz group and the irreducible representations of the Poincaré group, so that we could put those together to embed particles in fields.

It is now time to start quantising stuff. We will go through each spin in turn, as we did when constructing Lagrangians.

I forgot to emphasise this here!

Quantum fields are operator-valued functions of spacetime that transform under irreducible representations of the Lorentz group.

Spin 0

Schwartz 2.3

Tong 2.1, 2.2 ; P+S 2.3

The classical, relativistic scalar field satisfies the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0$$

which we can solve by Fourier transforming

$$\phi(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} e^{i\bar{p}\cdot\bar{x}} \tilde{\phi}(\bar{p},t)$$

satisfies $\left(\frac{\partial^2}{\partial t^2} + \bar{p}^2 + m^2\right)\tilde{\phi}(\bar{p},t) = 0$

The equation for a simple harmonic oscillator with frequency $\omega_{\bar{p}} = \sqrt{\bar{p}^2 + m^2}$!!

By moving to momentum space, we have decoupled the degrees of freedom in the scalar field: for each value of the momentum, the real scalar field is just a simple harmonic oscillator! Put another way, the general solution to the Klein-Gordon equation is a "linear superposition of simple harmonic oscillators, each vibrating at a different frequency with a different amplitude" (Tong, p. 22).

To quantise this scalar field, then, all we have to do is quantise each simple harmonic oscillator separately.

In our review, we saw that the harmonic oscillator Hamiltonian can be written as

see page (14) or any quantum mechanics textbook

$$\hat{H} = \omega\left(\hat{a}_-\hat{a}_+ + \tfrac{1}{2}\right) = \omega\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right)$$

more usual notation in QFT

We populate states by repeated application of the raising operator on the vacuum

$$|n\rangle = (\hat{a}^\dagger)^n|0\rangle$$

defined by $a|0\rangle = 0$

and we know

$$\hat{H}|n\rangle = (n + \tfrac{1}{2})\omega|n\rangle$$

note $\hat{H}|0\rangle = \frac{\omega}{2}|0\rangle$ is the "vacuum energy" or "zero point energy"

Now we apply our quantisation of each simple harmonic oscillator to the scalar field

$$\phi(\bar{x}) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\bar{p}}}} \left(a_{\bar{p}} e^{i\bar{p}\cdot\bar{x}} + a_{\bar{p}}^{\dagger} e^{-i\bar{p}\cdot\bar{x}} \right)$$

The conjugate momentum field is then

note analogy with quantum mechanics

$$\pi(\bar{x}) = \dot{\phi}(\bar{x})$$

$$= -i \int \frac{d^3\bar{p}}{(2\pi)^3} \sqrt{\frac{\omega_{\bar{p}}}{2}} \left(a_{\bar{p}} e^{i\bar{p}\cdot\bar{x}} - a_{\bar{p}}^{\dagger} e^{-i\bar{p}\cdot\bar{x}} \right)$$

$x = \frac{1}{\sqrt{2\omega}}(a + a^{\dagger})$

$p = \frac{1}{i}\sqrt{\frac{\omega}{2}}(a - a^{\dagger})$

Note :

Q: how do we know the coefficients are $a_{\bar{p}}$ and $a_{\bar{p}}^{\dagger}$ not $a_{\bar{p}}$ and $b_{\bar{p}}$, say?

1. we have <u>many</u> quantum mechanical systems, all at the same time, one for each momentum
2. we interpret the n^{th} excitation of the $\bar{p}$ harmonic oscillator as <u>n particles of momentum $\bar{p}$</u>

By analogy with quantum mechanics, we impose the commutation relation

$$\boxed{\left[a_{\bar{k}}, a_{\bar{p}}^{\dagger} \right] = (2\pi)^3 \delta^{(3)}(\bar{p} - \bar{k})}$$

← compare $[a, a^{\dagger}] = 1$ in quantum mechanics

note this is an <u>equal-time commutation relation</u>

The creation operator creates states with momentum $\bar{p}$

$$a_{\bar{p}}^{+}|0\rangle = \frac{1}{\sqrt{2\omega_{\bar{p}}}}|\bar{p}\rangle$$

vacuum state, defined by $a_{\bar{p}}|0\rangle = 0$
and normalised by $\langle 0|0\rangle = 1$

and our momentum states satisfy

$$\langle \bar{p}|\bar{k}\rangle = 2\sqrt{\omega_p \omega_k}\,\langle 0|a_p a_k^{+}|0\rangle$$

$$= 2\sqrt{\omega_p \omega_k}\,\langle 0|\left((2\pi)^3 \delta^3(\bar{p}-\bar{k}) + a_k^{+} a_p\right)|0\rangle$$

$$= 2(2\pi)^3 \omega_p\, \delta^3(\bar{p}-\bar{k})$$

$a_p|0\rangle = 0$

Note that

$$|\bar{k}\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2\omega_p}\, 2(2\pi)^3 \omega_p\, \delta^{(3)}(\bar{p}-\bar{k})\,|\bar{p}\rangle$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2\omega_p}\, |\bar{p}\rangle\langle \bar{p}|\bar{k}\rangle$$

$$\Rightarrow \mathbb{1} = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2\omega_p}\, |\bar{p}\rangle\langle \bar{p}|$$

We also have

$$\langle \bar{p}|\phi(\bar{x})|0\rangle = \langle 0|\sqrt{2\omega_p}\, a_p \int \frac{d^3\bar{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{i\bar{k}\cdot\bar{x}} + a_k^{+} e^{-i\bar{k}\cdot\bar{x}}\right)|0\rangle$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3} \sqrt{\frac{\omega_p}{\omega_k}} \left(e^{i\bar{k}\cdot\bar{x}} \langle 0|a_p a_k|0\rangle + e^{-i\bar{k}\cdot\bar{x}} \langle 0|a_p a_k^{+}|0\rangle\right)$$

$$= e^{-i\bar{p}\cdot\bar{x}}$$

$\phi(\bar{x})$ creates a particle at position $\bar{x}$

note analogy with $\langle \bar{p}|\bar{x}\rangle = e^{-i\bar{p}\cdot\bar{x}}$!

A note on normalisations: Schwartz says "We thus write $|\bar{p}\rangle$, $|p^{\mu}\rangle$ and $|p\rangle$ interchangeably" (page 21), but Tong defines

$$|p\rangle = \sqrt{2E_{\bar{p}}}\,|\bar{p}\rangle = \sqrt{2E_{\bar{p}}}\,a^{\dagger}_{\bar{p}}|0\rangle$$

with normalisation

$$\langle p|q\rangle = (2\pi)^3 \, 2E_{\bar{p}}\, \delta^{(3)}(\bar{p}-\bar{q})$$

and identity operator

$$\mathbb{1} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\bar{p}}} |p\rangle\langle p|$$

This is the normalisation of $|\bar{p}\rangle$ (and $|p\rangle$) in Schwartz! And in P+S

Tong's states $|\bar{p}\rangle$, on the other hand, satisfy

$$\langle \bar{p}|\bar{q}\rangle = (2\pi)^3 \delta^{(3)}(\bar{p}-\bar{q})$$

← not Lorentz invariant see P+S page 22

with

$$\mathbb{1} = \underbrace{\int \frac{d^3p}{(2\pi)^3}}\,\underbrace{|\bar{p}\rangle\langle\bar{p}|}$$

separately Lorentz non-invariant, but together invariant!

Note:

- $E_{\bar{p}}\,\delta^{(3)}(\bar{p}-\bar{q})$ <u>is</u> Lorentz invariant ← proof page 23 P+S
- $\int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E_{\bar{p}}}$ <u>is</u> Lorentz invariant ← proof page 23 P+S or Tong page 32

Commutation relations

The commutation relations for the creation and annihilation operators lead to commutation relations for the field operators themselves

$$\boxed{\begin{aligned}[\phi(\bar{x}), \phi(\bar{y})] &= [\pi(\bar{x}), \pi(\bar{y})] = 0 \\ [\phi(x), \pi(\bar{y})] &= i\,\delta^{(3)}(\bar{x}-\bar{y})\end{aligned}}$$

↑ again, equal-time commutation relations

↖ compare $[\hat{x}, \hat{p}] = i$ for quantum mechanics

Let's see how this works:

$$[\phi(\bar{x}), \pi(\bar{y})] = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \int \frac{d^3\bar{q}}{(2\pi)^3} \left(-i\sqrt{\frac{\omega_q}{2}}\right)$$

$$\times \left([a_p^+, a_q] e^{-i\bar{p}\cdot x + i\bar{q}\cdot y} - [a_p, a_q^+] e^{i\bar{p}\cdot\bar{x} - i\bar{q}\cdot\bar{y}} \right)$$

$$= -\frac{i}{2} \int \frac{d^3\bar{p}\, d^3\bar{q}}{(2\pi)^6} \sqrt{\frac{\omega_q}{\omega_p}} \left(-(2\pi)^3 \delta^{(3)}(\bar{p}-\bar{q}) \times e^{-i\bar{p}\cdot x + i\bar{q}\cdot\bar{y}} - (2\pi)^3 \delta^{(3)}(\bar{p}-\bar{q}) \times e^{i\bar{p}\cdot x - i\bar{q}\cdot\bar{y}} \right)$$

$$= \frac{i}{2} \underbrace{\int \frac{d^3 p}{(2\pi)^3} \left(e^{-i\bar{p}\cdot(\bar{x}-\bar{y})}\right.}_{=\delta^{(3)}(\bar{x}-\bar{y})} \left. + e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \right)$$

$$= i\,\delta^{(3)}(\bar{x}-\bar{y})$$

#

The proof of $[\phi(\bar{x}), \phi(\bar{y})] = 0$ is on page 24 of Schwartz.

What does the Hamiltonian look like in terms of our creation and annihilation operators?

Remember that

$$H = \frac{1}{2}\int d^3\bar{x}\left[\pi^2 + (\bar{\nabla}\phi)^2 + m^2\phi^2\right]$$

and then plug in our expressions to obtain

$$H = \frac{1}{2}\int d^3\bar{x}\int\frac{d^3\bar{p}\,d^3\bar{q}}{(2\pi)^6}\Big\{-\frac{\sqrt{\omega_{\bar{p}}\omega_{\bar{q}}}}{2}\left(a_{\bar{p}}e^{i\bar{p}\cdot\bar{x}} - a^+_{\bar{p}}e^{-i\bar{p}\cdot\bar{x}}\right)\times\left(a_{\bar{q}}e^{i\bar{q}\cdot\bar{x}} - a^+_{\bar{q}}e^{-i\bar{q}\cdot\bar{x}}\right)$$
$$+\frac{1}{2\sqrt{\omega_{\bar{p}}\omega_{\bar{q}}}}\left(i\bar{p}\,a_{\bar{p}}e^{i\bar{p}\cdot\bar{x}} - i\bar{p}\,a^+_{\bar{p}}e^{-i\bar{p}\cdot\bar{x}}\right)\left(i\bar{q}a_{\bar{q}}e^{i\bar{q}\cdot\bar{x}} - i\bar{q}a_{\bar{q}}e^{-i\bar{q}\cdot\bar{x}}\right)$$
$$+\frac{m^2}{2\sqrt{\omega_{\bar{p}}\omega_{\bar{q}}}}\left(a_{\bar{p}}e^{i\bar{p}\cdot\bar{x}} + a^+_{p}e^{-i\bar{p}\cdot\bar{x}}\right)\left(a_{\bar{q}}e^{i\bar{q}\cdot\bar{x}} + a^+_{\bar{q}}e^{-i\bar{q}\cdot\bar{x}}\right)\Big\}$$

Note $\int d^3\bar{x}\, e^{i\bar{p}\cdot\bar{x} - i\bar{q}\cdot\bar{x}} = (2\pi)^3\delta^{(3)}(\bar{p}-\bar{q})$

$$= \frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\bar{p}}}\Big\{\left(-\omega_{\bar{p}}^2 + \bar{p}^2 + m^2\right)\left(a_{\bar{p}}a_{-\bar{p}} + a^+_{\bar{p}}a^+_{-\bar{p}}\right) + \left(\omega_{\bar{p}}^2 + \bar{p}^2 + m^2\right)\left(a_{\bar{p}}a^+_{\bar{p}} + a^+_{\bar{p}}a_{\bar{p}}\right)\Big\}$$

Note $\omega_{\bar{p}}^2 = \bar{p}^2 + m^2$

$$= \frac{1}{2}\int\frac{d^3\bar{p}}{(2\pi)^3}\omega_{\bar{p}}\left(a_{\bar{p}}a^+_{\bar{p}} + a^+_{\bar{p}}a_{\bar{p}}\right)$$

$[a_{\bar{p}}, a^+_{\bar{p}}] = (2\pi)^3\delta^{(3)}(\bar{0})$

$$= \int\frac{d^3\bar{p}}{(2\pi)^3}\omega_{\bar{p}}\left(a^+_{\bar{p}}a_{\bar{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(\bar{0})\right)$$

So the Hamiltonian is divergent?!