Spin 1

Recall that the representations of the Poincaré group with spin j have $2j+1$ degrees of freedom if $m>0$ and two degrees of freedom if $m=0$. Let's look at each of these cases in turn.

Massive spin 1 particles

← e.g. $W^\pm, Z^0$ bosons

A massive ($m>0$) spin 1 particle has $2\cdot 1+1=3$ degrees of freedom. Therefore the "smallest" field we can embed this in is a vector field (with four components). You might wonder - what about the fourth degree of freedom? There are two points of view on the solution: the first is to recognise that the four-dimensional vector representation of the Lorentz group can be decomposed into a three-dimensional representation of SO(3) and a one-dimensional representation; and the second is to require a reasonable Lagrangian, which will entail that the extra degree of freedom does not contribute to physical observables.

more specifically a positive energy density requirement

The Lagrangian that gives a positive energy density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$$ ← the "Proca Lagrangian"

$$= \frac{1}{2}\left[A_\mu \partial^2 A^\mu - A_\mu \partial^\mu \partial^\nu A_\nu\right] + \frac{1}{2} m^2 A_\mu A^\mu$$

↑ This can be "derived" via

$$\mathcal{L} = a A_\mu \partial^2 A^\mu + b A_\mu \partial^\mu \partial^\nu A_\nu + \frac{m^2}{2} A_\mu A^\mu$$

$$\Rightarrow 2a\partial^2 A_\mu + 2b\, \partial_\mu \partial^\nu A_\nu + m^2 A_\mu = 0$$

$$\overset{\partial^\mu}{\Rightarrow} \left[2(a+b)\partial^2 + m^2\right]\partial^\mu A_\mu = 0$$

So we can eliminate one degree of freedom through $\partial^\mu A_\mu = 0$ by requiring $a = -b$ and $m > 0$.

↓ equations of motion are

$$(\partial^2 + m^2) A^\mu = 0$$

$$\partial_\mu A^\mu = 0$$

↑ this is interpreted as a constraint on A^μ

Schwarz shows that the energy density can be written in the form

↖ Schwarz p. 116

$$\mathcal{E} = T_{00} = \frac{1}{2}\left(\bar{E}^2 + \bar{B}^2\right) + \frac{m^2}{2}\left(A_0^2 + \bar{A}^2\right) > 0$$

Of more interest to us are the solutions to the Proca equations - the equations of motion - which we can write in the form:

$$A^\mu(x) = \sum_{j=1}^{3} \int \frac{d^3\bar{p}}{(2\pi)^3}\, \tilde{a}_j(\bar{p})\, \epsilon_j^\mu(p) e^{ip\cdot x}$$

↑ basis vectors

Any orthogonal basis vectors would do, but the most useful choice is one that enforces

$$\partial_\mu A^\mu = 0$$

$\uparrow$ one choice that achieves this is to pick

$$p_\mu \epsilon_j^\mu(p) = 0$$

For fixed $p^2 = m^2$, there are three independent solutions to this equation.

Q: why 3 not 4?

As an example, let's take

$$p^\mu = (E, 0, 0, p^z) \qquad \text{with} \quad E^2 = m^2 + p^{z2}$$

then our equation corresponds to

$$E\epsilon_j^0 - p^z\epsilon_j^3 = 0 \quad \Rightarrow \quad \text{choose} \quad \epsilon_j^3 = \frac{E}{p^z}\epsilon_j^0$$

What about the other components? We are free to choose these, so two natural (orthogonal) choices are

$$\left.\begin{aligned} \epsilon_1^\mu &= (0,1,0,0) \\ \epsilon_2^\mu &= (0,0,1,0) \end{aligned}\right\} \quad \text{satisfy} \quad \epsilon_\mu^*\epsilon^\mu = -1$$

← transverse polarisations

Then, choosing to normalise our third vector similarly, we have

$$\epsilon_3^\mu = \frac{1}{m}(p^z, 0, 0, E)$$

↖ also satisfies $\epsilon_\mu^*\epsilon^\mu = -1$

↖ longitudinal polarisation

Massless spin 1 particles

Schwartz 8.2.3

The correct massless spin-1 Lagrangian is exactly what you think it is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

but we cannot just take the $m \to 0$ limit of the previous discussion

for example $m^2(\partial_\mu A^\mu) = 0$ is no longer a constraint on A^μ if $m = 0$

First we note that this Lagrangian exhibits gauge invariance

Not surprising, because this is Lagrangian for a photon, and we know from electromagnetism that there should be a gauge symmetry at work in any theory of light

The transformation

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x)$$

Note this is a local symmetry!

leaves the Lagrangian invariant, so the fields $A^\mu(x)$ and $A'^\mu(x)$ are physically equivalent

$$\begin{aligned} F^{\mu\nu}(x) &\to F'^{\mu\nu}(x) \\ &= \partial^\mu(A^\nu + \partial^\nu\alpha) - \partial^\nu(A^\mu + \partial^\mu\alpha) \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu\partial^\nu\alpha - \partial^\nu\partial^\mu\alpha \\ &= F^{\mu\nu}(x) \end{aligned}$$

The equations of motion for this Lagrangian are

$$\partial^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$$

which we can write out explicitly as

$$-\bar{\nabla}^2 A^0 + \partial_t \bar{\nabla}\cdot\bar{A} = 0 \qquad (*)$$

$$\partial^2 A^i - \partial^i(\partial_t A^0 - \bar{\nabla}\cdot\bar{A}) = 0 \qquad (**)$$

We can make our lives easier by gauge fixing - choosing a convenient $\alpha(x)$. In particular, since

$$\bar{\nabla}\cdot\bar{A} \to \bar{\nabla}\cdot\bar{A}' = \bar{\nabla}\cdot\bar{A} + \bar{\nabla}^2\alpha$$

"Coulomb gauge"

we might as well pick α so that $\bar{\nabla}\cdot\bar{A} = 0$. Then the first equation $(*)$ tells us

$$\bar{\nabla}^2 A^0 = 0$$

But Coulomb gauge allows us to set $A^0 = 0$!

To see this, note Coulomb gauge is preserved under

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu\alpha$$

provided $\bar{\nabla}^2\alpha = 0$, so there is a remaining symmetry

$$A^0 \to A'^0 = A^0 + \partial_t\alpha$$

that we can use to eliminate A^0

ie "hidden" in the time dependence of $\alpha(x)$

This leaves us with three degrees of freedom. To eliminate the last one, we note that

$$\partial^2 \bar{A} = 0 \qquad \leftarrow \text{in Coulomb gauge}$$

which has general solution

$$A^\mu(x) = \sum_{j=1}^{3} \int \frac{d^3\bar{p}}{(2\pi)^3}\, \tilde{a}(\bar{p})\, \epsilon_j^\mu(p)\, e^{ip\cdot x}$$

Imposing the equations of motion, this gives us

$$p^2 = 0$$

and, from our choice of Coulomb gauge,

$$\epsilon_j^i(p)\, p^i = 0 \qquad \text{and} \qquad \epsilon_j^0 = 0$$

We can choose $p^\mu = (E, 0, 0, E)$ and then these equations are satisfied by

$$\left.\begin{aligned} \epsilon_1^\mu &= (0,1,0,0) \\ \epsilon_2^\mu &= (0,0,1,0) \end{aligned}\right\} \text{ we have two transverse polarisations!}$$

$\uparrow$ linear polarisation

$\uparrow$ another option is $\epsilon^\mu_{R/L} = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$ $\leftarrow$ transverse polarisation

To summarise :

- massive spin-1 particle
  - $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \quad \Rightarrow \quad (\partial^2 + m^2) A^\mu = 0$
    $\partial_\mu A^\mu = 0$
  - 3 polarisations, eg $\epsilon_1^\mu = (0,1,0,0)$
    $\epsilon_2^\mu = (0,0,1,0)$
    $\epsilon_3^\mu = \frac{1}{M}(p^3, 0, 0, E)$

- massless spin-1 particle
  - $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \Rightarrow \quad \partial^2 A^\mu - \partial^\mu \partial^\nu A_\nu = 0$
    $\partial_\mu A^\mu = 0$
  - gauge invariance leaves us with 2 polarisations
    eg $\epsilon_1^\mu = (0,1,0,0)$ or $\epsilon_{R/L} = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$
    $\epsilon_2^\mu = (0,0,1,0)$

Higher spin particles

We could carry on and enumerate all the higher spin particles, but that would take us far off track. As far as we know, there are no experimentally observed fundamental particles of higher spin, so if they discover these, we can always come back to this topic.

The "graviton" would be a massless spin 2 particle but a quantum theory of gravity is beyond this cou