Tong 4.6

Continuous symmetries

- spacetime translations $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$

$$\Rightarrow \delta\psi = \epsilon^\mu \partial_\mu \psi$$
$$\Rightarrow T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\mathcal{L}$$

the energy-momentum tensor

The equations of motion are already implied in the (derivation of the) conserved current, so we can choose $\mathcal{L} = 0$ and

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi$$

Then the total energy is

$$\begin{aligned} E &= \int d^3\bar{x}\, T^{00} \\ &= i\int d^3\bar{x}\, \bar{\psi}\gamma^0\partial^0\psi \\ &= \int d^3\bar{x}\, \bar{\psi}\,(-i\gamma^i\partial_i + m)\psi \end{aligned}$$

- Lorentz transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\Rightarrow \delta\psi^\alpha = -\omega_{\mu\nu}\left(x^\nu\partial^\mu\psi^\alpha - \tfrac{1}{2}(\mathcal{J}^{\mu\nu}_{1/2})^\alpha{}_\beta\psi^\beta\right)$$
$$\Rightarrow (\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} + \bar{\psi}\gamma^\mu \mathcal{J}^{\rho\sigma}_{1/2}\psi$$

When we quantise, as we will see, $\mathcal{J}^\mu$ becomes an operator and this gives rise to an internal angular momentum (ie. a particle with spin $\frac{1}{2}$!)

• Vector symmetry $\quad \psi \to \psi' = e^{-i\alpha}\psi$

$\Rightarrow \delta\psi = -i\delta\alpha\,\psi$

N.B. $\partial_\mu J^\mu_V = (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu\partial_\mu\psi = im\bar\psi\psi - im\bar\psi\psi = 0$

$\Rightarrow J^\mu_V = \bar\psi\gamma^\mu\psi$

this is the conserved <u>vector current</u>

The corresponding conserved charge is

$$Q = \int d^3\bar{x}\,\bar\psi\gamma^0\psi = \int d^3\bar{x}\,\psi^*\psi = \int d^3\bar{x}\,|\psi|^2$$

electric charge (or, equivalently, particle #)

• Axial symmetry $(m=0)$ $\quad \psi \to \psi' = e^{i\alpha\gamma^5}\psi,\ \bar\psi \to \bar\psi' = \bar\psi e^{i\alpha\gamma^5}$

N.B. $\partial_\mu J^\mu_A = (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5(\partial_\mu\psi) = 2im\bar\psi\gamma^5\psi$

$\Rightarrow \delta\psi = i\delta\alpha\gamma^5\psi,\ \delta\bar\psi = +i\delta\alpha\ \bar\psi\gamma^5$

note $e^{-i\alpha\gamma^5}\gamma^0 = \gamma^0 e^{i\alpha\gamma^5}$

$\Rightarrow J^\mu_A = \bar\psi\gamma^\mu\gamma^5\psi$

This is one of the most interesting symmetries, b/c it is <u>anomalous</u>: it is a symmetry of the classical theory, but <u>not</u> of the quantum theory!

Here $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$(\gamma^5)^\dagger = \gamma^5$

$(\gamma^5)^2 = \mathbb{1}$

$\{\gamma^5, \gamma^\mu\} = 0$

* Quick Quiz